Math 565a - Homework 3

1. Consider the simple symmetric random walk S_n on $\mathbb{Z}^d$. The goal of this problem is to show it is transient in three or more dimensions by showing that $\sum_n P(S_n = 0) < \infty$.
(a) Let X_n be the steps. Compute the characteristic function of a single step:

$$\phi(t) = E[\exp(it \cdot X_1)], \quad t \in \mathbb{R}^d$$

(b) For $0 < r < 1$, show that

$$\sum_{n=0}^{\infty} r^n P(S_n = 0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{1}{1 - r\phi(t)} \, d^d t$$

(c) Let $r \to 1$ in the above to show that $\sum_n P(S_n = 0) < \infty$ if $d \geq 3$.

2. Prove the following proposition that was stated in class.
(a) If X_n is a renewal sequence then the sequence of waiting times W_n is independent and indentically distributed.
(b) Given an i.i.d. sequence W_n, there is a renewal sequence X_n whose waiting times are W_n.

3. Let S_n be a random walk in $\mathbb{R}^d$ which takes values in a discrete set. Let $x \in \mathbb{R}^d$ be a possible site, i.e, a site with $P(\exists n \, s.t. \, S_n = x) > 0$. Let $X_n = 1$ if $S_n = x$ and $X_n = 0$ if $S_n \neq x$. Show that X_n is a delayed renewal sequence.

4. Let X_n be a renewal sequence with potential measure U, waiting time distribution R, and renewal measure N.
(a) Prove that if $U(\mathbb{Z}^+) < \infty$, then $N(\mathbb{Z}^+)$ is a geometric random variable with mean $U(\mathbb{Z}^+)$ and $U(\mathbb{Z}^+) = 1/R(\infty)$.
(b) Prove that if $U(\mathbb{Z}^+) = \infty$, then $N(\mathbb{Z}^+) = \infty$ a.s. and $R(\infty) = 0$.

5. Let S_n be a random walk in $\mathbb{R}$. Recall that the time n is a strict ascending ladder time if $S_n > S_m$ for $m = 0, 1, 2, \cdots, n-1$. Prove that the strict ascending ladder times are the regenerative set of a renewal process.

6. Consider a one dimensional random walk and two renewal processes associated with it : the strict ascending ladder times and the weak descending ladder times. Prove that either both of these renewal processes are null recurrent or one is transient and the other is positive recurrent. Hint: Use the equation we derived in class relating the generating functions of the waiting time distribution for these two renewal processes.

7. Let X_n be an aperiodic renewal process. Let $\epsilon_0, \epsilon_1, \cdots, \epsilon_r \in \{0, 1\}$ with $\epsilon_0 = 1$. Find

$$\lim_{n\to\infty} P(X_{n+j} = \epsilon_j, 0 \leq j \leq n)$$

and justify your answer. Hint: condition on the event $X_n = 1$. For more of a challenge, do not assume $\epsilon_0 = 1$. (This is optional.)