Math 565a - Homework 4 (corrected March 9)

1. Let X_n be an i.i.d. sequence with finite first and second moments. Let $S_n = \sum_{k=1}^n X_k$, so S_n is a random walk. We saw in class that S_n is a martinagle if and only if $EX_n = 0$. When the mean is not zero, we can still get a related martingale, namely $S_n - n\mu$ where $\mu = EX_1$. Find c_n and d_n so that

$$Y_n = S_n^2 + c_n S_n + d_n$$

is a martingale. (c_n and d_n will depend on n and the first and second moments of X_1.)

2. Let X_n and Y_n be $\mathcal{F}_n$ submartingales. Let $Z_n = \max\{X_n, Y_n\}$. Prove Z_n is an $\mathcal{F}_n$ submartingale.

3. Let X_n and Y_n be $\mathcal{F}_n$ supermartingales. Let τ be an $\mathcal{F}_n$ stopping time which is finite almost surely. Assume $X_\tau \geq Y_\tau$ a.s. Define

$$Z_n = \begin{cases} X_n & \text{if } n \leq \tau \\ Y_n, & \text{if } n > \tau \end{cases}$$

Prove that Z_n is an $\mathcal{F}_n$ supermartingale.

4. (Polya urn) An urn starts with r red balls and g green balls. We draw a ball at random and replace it with m balls of the same color. We repeat this process. Let X_n be the number of red balls at time n, Y_n the number of green balls at time n and

$$R_n = \frac{X_n}{X_n + Y_n}$$

Show that R_n is a martingale.

5. (a) Let S_n be a random walk whose steps are integrable and have a non-negative mean. So S_n is a submartingale. Find the Doob decomposition of S_n.
(b) Let $B_m \in \mathcal{F}_m$. Assume that the B_m are independent of $\mathcal{F}_{m-1}$. (This was missing in the original version of the problem, and in my first corrected version I mistakenly said the B_m are independent of each other.) Define

$$X_n = \sum_{k=1}^n 1_{B_k}$$

Show X_n is a submartingle and find the Doob decomposition of X_n.

6. **Note : this problem has been significantly changed since the original version.**

(a) Let X_n be i.i.d. and $S_n = \sum_{k=1}^n X_k$. So S_n is a random walk. Assume that $E[\exp(\alpha X_1)] < \infty$ for all real α. Define for real α

$$Z_n = \exp[\alpha S_n + nc]$$

Show that for a suitably chosen constant c, Z_n is a martingale.

(b) Consider the simple symmetric one-dimensional random walk. Let $a \in \mathbb{Z}$. Define

$$\tau = \inf\{n : S_n = a\}$$

Find the moment genertating function of τ, i.e., find $E[\exp(t\tau)]$ for $t < 0$.