Math 565a - Homework 5

1. In some problems the process itself is not a Markov process, but one can construct a closely related Markov process by "enlarging the state space." Then you can apply the theory of Markov processes to your problem. Here is a simple example.

We work on the lattice $\mathbb{Z}^2$. We generate a sample of the process as follows. Take $X_0 = 0$. Having generated $X_1, X_2, \cdots, X_n$, consider the four lattice sites that are a distance 1 from X_n. One of them is X_{n-1}. Take X_{n+1} to be one of the other three with equal probability 1/3. The filtration is the natural one: $\mathcal{F}_n = \sigma(X_1, X_2, \cdots, X_n)$. One can describe this process as a random walk that is not allowed to "backtrack."

(a) Show that X_n is not a Markov process.

(b) Let Y_n be the random vector (X_{n-1}, X_n) for $n = 1, 2, \cdot$. The filtration is the same as before, or equivalently $\mathcal{F}_n = \sigma(Y_1, Y_2, \cdots, Y_n)$. For the original process the state space is $\mathbb{Z}^2$. for the Y_n process it is $\mathbb{Z}^2 \times \mathbb{Z}^2$. Show that Y_n is a Markov process.

2. (Durrett 1.6, p. 284) Fix a positive integer N. Let $\xi_1, \xi_2, \cdots$ be i.i.d. where each ξ_n is uniformly distributed on $\{1, 2, \cdots, N\}$. Consider the set $\{\xi_1, \xi_2, \cdots, \xi_n\}$ of values that have occurred by time n. It is a subset of $\{1, 2, \cdots, N\}$ and will eventually have repetitions. Let X_n be the number of distinct values that appear in this set. Show that this is a Markov process and find the transition probabilities.

3. (Durrett 1.7, p. 285) Let S_n be a simple random walk on $\mathbb{Z}$ with $S_0 = 0$. Let $M_n = \max\{S_k : 0 \leq k \leq n\}$. Show that M_n is not a Markov process.

4. (Durrett 1.8, p. 285) Let $\theta, U_1, U_2, \cdots$ be independent with each random variable uniformly distribution on $[0, 1]$. Define X_n by

$$X_n = \begin{cases} +1 & \text{if } U_n < \theta \\ -1 & \text{if } U_n \geq \theta \end{cases}$$

Then let $S_n = \sum_{k=1}^n X_k$. We can think of this as a sort of random walk. We first choose a "bias" θ. We then flip a coin with probability θ of heads and probability $1 - \theta$ of tails to generate the random walk. (The bias does not change as we flip the coin.) Show that S_n is a Markov process which is not time homogeneous.

5. (Durrett 2.10, p. 290) Consider a Markov process with a countable state space S. Let $A \subset S$ such that $S \setminus A$ is finite. Assume that for each $x \notin A$,

$P_x(\tau_A < \infty) > 0$. Let

$$\tau_A = \inf\{n \geq 0 : X_n \in A\}$$

and let $g(x) = E_x \tau_A$. Let $p(x,y)$ be the transition probabilities. Show that

$$g(x) = 1 + \sum_{y \in S} p(x,y) g(y)$$

for $x \notin A$.

6. (the ballot problem) Let Y_j be i.i.d. with values in $\{0, 1, 2, \cdots\}$. Let $S_n = Y_1 + \cdots + Y_n$. Fix N. The goal of this problem is to show

$$P(S_j < j, 1 \leq j \leq N | S_N) = (1 - \frac{S_N}{N})^+$$

Note that if $S_N \geq N$, the equation is trivially true. So you can assume $S_N < N$. I showed in class that $X_n = S_n/n$ is a backwards martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_k, k \geq n)$. Define $\tau = \max\{k : S_k \geq k, 1 \leq k \leq N\}$ if this set is non-empty. If the set is empty we define $\tau = 1$. We want to apply the optional sampling theorem to the backwards martingale and τ. If we go backward in time X_n is a martingale.

(a) Show that τ is a stopping time if we go backwards in time. (Part of the problem is to figure out what this means.)

(b) Show that $X_\tau = 1_{G^c}$ where G is the event $\{S_j < j, 1 \leq j \leq N\}$.

(c) Apply the optional sampling theorem to get the result stated above.