Math 565a - Homework 6

1. (easy) Consider the branching process and assume that the probability an individual has no children is not zero. Show that all states but one are transient. What is the one recurrent state?

2. (Birth and death chain continued) Consider a birth and death chain in which $p(x, x+1) = p_x$, $p(x, x-1) = q_x$, $p(x, x) = 1 - p_x - q_x$ and all other transitions have probability zero. We take $q_0 = 0$. We assume $p_x > 0$ for $x \geq 0$ and $q_x > 0$ for $x \geq 1$ so the chain is irreducible. In class we showed

$$P_x(\tau_a < \tau_b) = \frac{\sum_{y=x}^{b-1} \frac{q_y}{p_y} \frac{q_{y-1}}{p_{y-1}} \cdots \frac{q_{a+1}}{p_{a+1}}}{\sum_{y=a}^{b-1} \frac{q_y}{p_y} \frac{q_{y-1}}{p_{y-1}} \cdots \frac{q_{a+1}}{p_{a+1}}}$$

where $\tau_z = \inf\{n \geq 0 : X_n = z\}$ (In these sums the $y = a$ term is an empty product and so should be taken to be 1). Use this to prove that the chain is transient if and only if

$$\sum_{y=0}^{\infty} \frac{q_y}{p_y} \frac{q_{y-1}}{p_{y-1}} \cdots \frac{q_1}{p_1} < \infty$$

3. (Birth and death chain using martingales) For the birth and death chain, define

$$\phi(n) = 1 + \frac{q_1}{p_1} + \frac{q_1 q_2}{p_1 p_2} \cdots + \frac{q_1 q_2 \cdots q_{n-1}}{p_1 p_2 \cdots p_{n-1}}$$

(We take $\phi(0) = 0$ and $\phi(1) = 1$.) Show that $Y_n = \phi(X_n)$ is a martinagle and use it to derive the formula for $P_x(\tau_a < \tau_b)$ in the previous exercise.

4. (Durrett 4.2, p. 304) Let X_n be a renewal process. We defined a Markov chain called the renewal chain by $Y_n = \inf\{k > n : X_k = 1\} - n$, i.e., Y_n is the time to the next renewal. Assume that the waiting time is finite with probability one.
(a) Find a necessary and sufficient condition on the waiting time distribution for all the states to be recurrent.
(b) In class we showed that if x is a recurrent site, then the following is a stationary measure:

$$\mu_x(y) = E_x(\sum_{n=0}^{\tau-1} 1_{X_n = y})$$

where $\tau = \inf\{n \geq 1 : X_n = x\}$. Use this to find a stationary measure for the renewal Markov chain.
(c) Find a necessary and sufficient condition for your stationary measure to be normalizable.

5. (Watkins) Let X_n be a Markov process. Use the Markov property to show that if $A \in \sigma(X_0, X_1, \cdots, X_n)$ and $B \in \sigma(X_n, X_{n+1}, \cdots)$, then for any initial distribution α, we have

$$P_\alpha(A \cap B|X_n) = P_\alpha(A|X_n)P_\alpha(B|X_n)$$

In words, the past and future are conditionally independent given the present. Hint

$$P_\alpha(A \cap B|X_n) = E_\alpha[E_\alpha[1_A 1_B|\mathcal{F}_n]|X_n]$$

6. (Durrett 4.1, p. 302) Suppose that a Markov process has a stationary distribution π that is a probability measure. Assume that the state space is countable. Let X_n be the Markov process with initial distribution π. Fix a positive integer N and define a process Y_m for $0 \leq m \leq N$ by $Y_m = X_{N-m}$. Show that Y_m is a Markov process and find its transition probabilities in terms of those of the original process X_n.