Math 565a - Homework 7

1. (Durrett p. 293, 3.4) Use the strong Markov property to prove that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$.

2. (Durrett p. 299, 3.9) Let $p(x,y)$ be an irreducible transition matrix for a countable state space. A function f is superhamonic (with respect to p) if $f(x) \geq \sum_y p(x,y)f(y)$ for all x. Prove that the process is recurrent if and only if every non-negative superharmonic function is constant.

3. (Durrett p. 309, p. 4.10) In chess the board is an 8 by 8 grid of squares. The legal moves for a knight are to take two steps in one direction, then one step in a direction perpendicular to the first two. (Only moves that land on the board are allowed). A knight starts in the corner with no other pieces on the board and randomly picks his moves. Find the mean of the return time to the corner he starts from.

4. (Durrett p. 324, 5.9 and 5.10) In class we proved that if we have an irreducible, positive recurrent, aperiodic Markov process on a countable state space, then for any initial distribution the distribution of X_n converges in total variation norm to the stationary distribution. But the theorem did not say anything about the rate of convergence. The goal of this problem is to prove the convergence is exponentially fast if the state space is finite. So for both parts below assume that S is finite and $p(x,y)$ is irreducible and aperiodic. (Since S is finite these two properties imply it is positive recurrent.)
(a) Prove there is a single N so that $p^N(x,y) > 0$ for all x, y. (This follows easily from a result in class.)
(b) Let τ be the coupling time used in the proof of the theorem above. Prove that $P(\tau > n) \leq cr^n$ for some constants c, r with $r < 1$. Use this to conclude the convergence of the distribution of X_n to the stationary probability measure is exponentially fast.

5. In class we proved a theorem about the long time behavior of an irreducible, positive recurrent Markov process with a countable state space S for the case that the period d is greater than 1. The first part of the theorem gave the limit

$$\lim_{n\to\infty} p^{nd+j}(x,y)$$

Let μ be a probability measure on S. Suppose we start the process in μ, i.e., $P(X_0 = x) = \mu(x)$. Prove that there are probability measures $\pi_0, \pi_1, \cdots, \pi_{d-1}$

such that for each $j = 0, 1, \cdots d-1$, as $n \to \infty$, μp^{nd+j} converges to π_j in total variation norm. (If $|| \quad ||$ is the total variation norm, then $||\mu_1 - \mu_2|| = \sum_x |\mu_1(x) - \mu_2(x)|$.)

6. A commonly used Markov chain Monte Carlo algorithm is the Metropolis-Hastings algorithm. Let S be a countable set, π a probability measure on S. We assume that $\pi(x) > 0$ for all $x \in S$. Let $t(x, y)$ be a transition function on S. We do not assume any relation to π, although in applications the structure of t is often motivated by the structure of S and π. Define

$$\alpha(x, y) = \min\{\frac{\pi(y)t(y, x)}{\pi(x)t(x, y)}, 1\}$$

and then define a new transition matrix

$$p(x, y) = \alpha(x, y)t(x, y) + (1 - \alpha(x, y))\delta_{x,y}$$

where $\delta_{x,y}$ is 1 if $x = y$ and 0 if $x \neq y$. Show that π is a stationary measure for p.

Comments: In applications π is often of the form $\pi(x) = \mu(x)/N$ where $\mu(x)$ is a finite probability measure with an explicit, relatively simple formula for $\mu(x)$. The normalization factor N is usually impossibly difficulty to compute. However, the above only requires $\pi(y)/\pi(x) = \mu(y)/\mu(x)$. So you do not have to compute N. The formula for $p(x, y)$ can be described as follows. If we are in state x, we use the transition probabilities $t(x, y)$ to pick a "proposed" state y to jump to. We then compute $\alpha(x, y)$ and make the jump with probability $\alpha(x, y)$ ("accept the proposed move") and just stay in the state x with probability $1 - \alpha(x, y)$ ("reject the proposed move").

7. (Durrett p. 332, p. 6.11) Prove that a Harris chain with a stationary probability measure must be recurrent. Hint: We proved (or will prove) in class that $\bar{X}_n$ is recurrent if and only if $\sum_{n=1}^{\infty} \bar{p}^n(\alpha, \alpha) = \infty$. Use this fact and mimic the proof of the following theorem from class: For a countable state space if there is a stationary probability measure π then all states y with $\pi(y) > 0$ are recurrent.