Math 565b - Homework 3

1. Recall that we defined a function $x(t)$ on $\mathbb{Q}^+$ (the positive rationals) to be *regularizable* if its left and right hand limits both exist and are finite. In class I stated but did not prove

Proposition: Let $x(t) : \mathbb{Q}^+ \to \mathbb{R}$ be regularizable. Define

$$y(t) = \lim_{q \to t^+} x(q) \tag{1}$$

where q is restricted to rationals. Then $y(t)$ is an R-function, i.e., it is right continuous and its left hand limits exist.

Prove the proposition.

2. On the set of integrable random variables define a metric by

$$d(X, X') = \inf\{\epsilon : P(|X - X'| > \epsilon) < \epsilon\} \tag{2}$$

Prove that a sequence X_n of random variables converges to a random variable X in probability if and only if $d(X_n, X) \to 0$.

3. Let B_t be standard Brownian motion. Prove that $\{B_t^2 : 0 \leq 1\}$ is uniformly integrable. Hint: Use a martingale.

4. Let T be a stopping time. The σ-field $\mathcal{F}_T$ is the collection of events A such that for all $t \geq 0$, $A \cap \{T \leq t\}$ is in $\mathcal{F}_t$. Suppose that $\mathcal{F}_t$ is a right continuous filtration. Prove that if we replace $\{T \leq t\}$ by $\{T < t\}$ in this definition then we get the same σ-field $\mathcal{F}_T$.

5. Let B_t be standard Brownian motion, and $\mathcal{F}_t^B$ the filtration where $\mathcal{F}_t^B$ is generated by $\{B_s : 0 \leq s \leq t\}$. Recall that we proved that all the events in $\mathcal{F}_{t+}^B$ have probability 0 or 1. Let $f(t)$ be a function on $(0, \infty)$ such that $f(t) > 0$ for all $t > 0$. Define a random variable X by

$$X = \limsup_{t \to 0^+} \frac{B_t}{f(t)} \tag{3}$$

The lim sup always exists, but of course it may be ∞. Prove that the random variable X is a constant (possibly infinite).

6. Use the result from the previous problem to prove that

$$\limsup_{t \to 0^+} \frac{B_t}{\sqrt{t}} = \infty \tag{4}$$

with probability one. B_t is still a standard Brownian motion.

7. (Watkins, p. 19) Let B_t be standard Brownian motion. Let $a > 0$ and define $\tau = \inf\{t \geq 0 : |B_t| = a\}$ Using the martingale $B_t^2 - t$ you can show $E\tau = a^2$. Show that $E\tau^2 = 5a^2/3$. Hint: Show that $B_t^4 - 6tB_t^2 + 3t^2$ is a martingale. You can show it is a martingale by brute force, but a much shorter method is to use the exponential martingale.

8. (Watkins, p. 20) Let N_t be a Poisson process. Let τ_n be the hitting time of n,i.e., $\tau = \inf\{t : N_t = n\}$. Let $\sigma_n = \tau_n - \tau_{n-1}$. Prove that σ_n is an i.i.d. sequence with exponential distribution.

9. (Watkins, p. 21) State and prove a reflection principle for symmetric Levy processes and general stopping times. We stated it for Brownian motion at the start of the course. You can also find a statement for BM in Watkins notes on page 21.

10. (Durrett, p. 402) Show that

$$X_t = \frac{\exp(B_t^2/(1+t))}{\sqrt{1+t}} \tag{5}$$

is a martingale and use this to show

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log t}} \leq 1 \tag{6}$$

with probability one.