**Math 565b - Homework 4**

1. Consider the semigroup

$$T(t)f(x) = f(x+t) \tag{1}$$

Find the generator of $T(t)$. Take the Banach space to be $C_0(\mathcal{R})$.

2. Consider a generalized Poisson process with $Y_k = \pm 1$ with probability $1/2$. This is like a Poisson process in which instead of always going up by 1 when it jumps, the process either goes up or down by 1 (with equal probability) when it jumps. Recall that the generalized Poisson process $X_t$ is defined by

$$X_t = \sum_{k=1}^{N_t} Y_k \tag{2}$$

where $Y_k$ is iid and $N_t$ is a Poisson process with rate $\lambda$.
(a) Find the generator of the semigroup associated with this process.
(b) The theory we have developed shows that this generator must be a dissipative operator. Prove this directly using your answer to (a).

3. Let $X_t$ be a Brownian motion with $EX_t = \nu t$ and $var(X_t) = \sigma^2 t$ where $\nu$ and $\sigma^2$ are constants. (Note that you can obtain $X_t$ by taking $B_t$ to be a standard Brownian motion and letting $X_t = \sigma^2 B_t + \mu t$. ) Find the generator of the semigroup associated with this Markov process.

4. Consider a Gaussian process with mean zero and covariance $C(s,t)$. It is defined for $t \geq 0$. Show that the process is a Markov process if and only if the covariance satisfies:

$$C(s,u)C(t,t) = C(s,t)C(t,u) \tag{3}$$

for $0 \leq s < t < u$.

5. Let $B_t$ be standard Brownian motion. Let $T(t)$ be the associated semigroup and $R_\lambda$ its resolvent. Show that the resolvent is an integral operator, i.e.,

$$R_\lambda f(x) = \int_{-\infty}^{\infty} r_\lambda(x,y) f(y) dy \tag{4}$$

and

$$r_\lambda(x,y) = \frac{1}{\sqrt{2\lambda}} \exp(-\sqrt{2\lambda}|x-y|) \tag{5}$$

6. Let $P(t, x, dy)$ be a time homogeneous transition function. In particular, it satisfies the Chapman-Kolmogorov eq. Let $\alpha$ be a probability measure on $S$. For $0 < t_1 < t_2 < \cdots < t_n$, define the finite dimension distribution of $X_0, X_{t_1}, \cdots, X_{t_n}$ by

$$P(X_0 \in B_0, X_{t_1} \in B_1, \cdots, X_{t_n} \in B_n) = \int_{B_0} \int_{B_1} \cdots \int_{B_{n-1}}$$
$$P(t_n - t_{n-1}, x_{n-1}, B_n) P(t_{n-1} - t_{n-2}, x_{n-2}, dx_{n-1}) \cdots P(t_1, x_0, dx_1) \alpha(dx_0)$$

where $B_0, B_1, \cdots, B_n$ are measurable subsets of the state space $S$. Use the Daniell-Kolmogorov extension theorem to show there is a stochastic process with these finite dimensional distributions. Note that we proved in class that for a Markov process with transition function $P$, the finite dimensional distributions are given by the above equation.

7. Let $X_t$ be a Markov process with transition function $P(t, x, B)$. We let $S^\Delta = S \cup \{\Delta\}$ with the topology defined as we did in class. So if $S$ is compact, $\Delta$ is an isolated point and if $S$ is not compact, $S^\Delta$ is the one point compactification of $S$. Let $A \subset S$ be a Borel set. Let

$$\tau_A = \inf\{t : X_t \in A\} \tag{6}$$

Take an initial distribution $\alpha$ such that $\alpha(A) = 1$. Define a new process $Y_t$ by $Y_t = X_t$ for $t < \tau_A$ and $Y_t = \Delta$ for $t \geq \tau_A$. Show $Y_t$ is a Markov process. It is usually described as the process $X_t$ killed when it exits $A$.

8. For a topological space $X$, $D_X[0, \infty)$ denotes the space of functions from $[0, \infty)$ into $X$ which are right continuous and have left hand limits at all $t$. Let $S$ be a locally compact Hausdorff space. Let $D$ be a dense subset of $C_0(S)$. Let $x : [0, \infty) \to S$. Prove that $x \in D_S[0, \infty)$ if and only if $f(x) \in D_{\mathcal{R}}[0, \infty)$ for all $f \in D$.

9. Let $S$ be a locally compact, Hausdorff space which is separable. Let $D \subset C_0(S)$ be dense. Prove that $D$ has a *countable* subset which is still dense in $C_0(S)$.

10. Let $X_t$ be a Markov process with transition function $P(t, x, B)$. Let $f : S \to \mathcal{R}$ be a bounded random variable. Prove $t \to f(X_t)$ is right continuous from $[0, \infty)$ into $L^1(\Omega, P)$. (We needed this to apply the Doob regularity thm.)