Math 565b - Homework 5

1. Consider a birth and death process. As in class, λ_x is the rate for jumping from x to $x+1$. We argued (non-rigorusly) in class that if

$$\sum_x \frac{1}{\lambda_x} < \infty \tag{1}$$

then the process can have "explosions" where the process runs off to $+\infty$ in a finite amount of time. Prove that if this sum is infinite, then the process is conservative. Recall that this means there is a sequence of functions f_n in $C_0(S)$ with $\sup_n ||f_n|| < \infty$, $f_n \to 1$ pointwise and for each $t \geq 0$, $T(t)f_n \to 1$ pointwise.

2. The Ornstein-Uhlenbeck process is the Markov diffusion process with generator

$$A = \frac{\sigma^2}{2} \frac{d^2}{dx^2} - \rho x \frac{d}{dx} \tag{2}$$

Show that a normal distribution with mean zero is stationary for this process.

3. Let B_t be standard Brownian motion. Define $X_t = \exp(B_t)$. Find the generator of this Markov process. Note: This process is called geometric Brownian. It shows up in financial mathematics a lot. To check your answer you can find the generator of geometric BM in Watkins notes.

4. Let $T(t)$ be the semigroup of a Markov process. Suppose there is probability measure ν on the state space such that for all $f \in C_0(S)$ and all initial distributions α,

$$\lim_{t \to \infty} \int T(t) f \, d\alpha = \int f \, d\nu \tag{3}$$

Prove that ν is a stationary measure.

5. Let $T(t)$ be the semigroup of a Markov process. Suppose there is a probability measure ν on the state space and an increasing sequence of times t_n which converges to ∞ and such that for all $f \in C_0(S)$ and all initial distributions α,

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \int T(t) f \, d\alpha \, dt = \int f \, d\nu \tag{4}$$

Prove that ν is a stationary measure.

6. Consider standard Brownian motion. Let $J = (a, b)$. For $x \in J$ let τ_J be the exit time for J when the Brownian motion starts at x. Show that

$$E_x \, \tau_J = (x - a)(x - b), \quad x \in J \tag{5}$$

Thus the speed measure is Lebesgue measure.

7. Show that Brownian motion is on "natural scale." This means that

$$P_x(\tau_b < \tau_a) = \frac{x - a}{b - a} \tag{6}$$

for $x \in (a, b)$ where τ_y is the hitting time for y.

8. Prove that a diffusion process is on natural scale if and only if for all $a, b \in I$ we have

$$P_x(\tau_b < \tau_a) = \frac{1}{2} \tag{7}$$

when we take $x = (a + b)/2$.