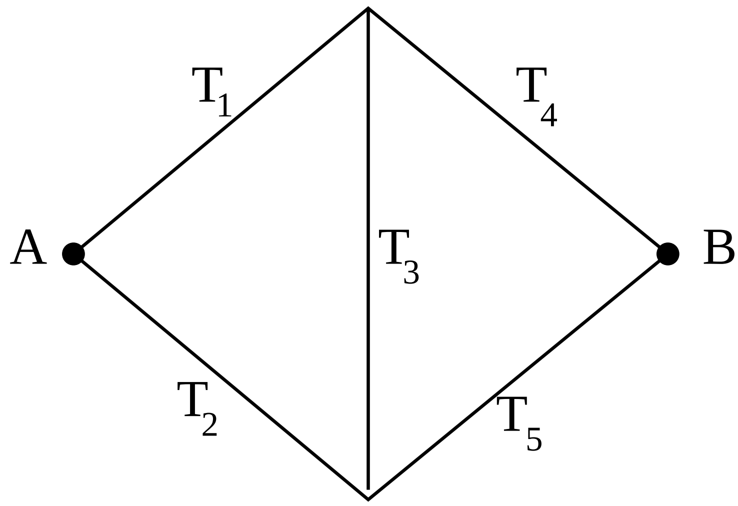
Monte Carlo Methods - Spring 16 - Homework 1

1. This is based on an example in the review article by Kroese. Consider the small network shown in the figure.



The random variables T_i are the times it takes to traverse that bond. The times T_i are uniformly distributed but with different ranges:
T_1 uniform on $[0, 1]$
T_2 uniform on $[0, 2]$
T_3 uniform on $[0, 3]$
T_4 uniform on $[0, 1]$
T_5 uniform on $[0, 2]$
We want to find the quickest path from A to B. Let X be the total time it takes to traverse this quickest path. We want to compute the mean $\mu = E[X]$.
(a) Using $N = 10,000$ samples, use direct Monte Carlo to estimate μ. Using your data estimate σ^2 and find a 95% confidence interval for your estimate. As a check on your simulation, the exact answer is know to be $\frac{1339}{1440} \approx 0.92986$.
(b) Repeat what you did in (a) 100 times and see how many times your confidence interval contains the exact value of μ.

2. (corrected 2/9) In class and in the notes I claimed that if you apply the delta method to find the variance of the estimator $\hat{\theta} = \overline{Y}/\overline{X}$ for $\theta = E[Y]/E[X]$, then the variance of $\hat{\theta}$ is approximately

$$var(\hat{\theta}) \approx \frac{1}{n} \frac{\sum_{i=1}^{n} (Y_i - \hat{\theta} X_i)^2}{n \overline{X}^2}$$

where $\overline{X}$ and $\overline{Y}$ are the sample means for X and Y and $\hat{\theta} = \overline{Y}/\overline{X}$. For the general estimator $\hat{\theta} = f(\hat{\mu}_1, \cdots, \hat{\mu}_d)$ we found from the delta method that the variance is $(\hat{\nabla} f, \hat{\Sigma}\hat{\nabla} f)/n$. Use this general result to derive the equation above, i.e., show

$$(\hat{\nabla} f, \hat{\Sigma}\hat{\nabla} f) = \frac{\sum_{i=1}^n (Y_i - \hat{\theta} X_i)^2}{n\overline{X}^2}$$

3. Let $X_1, \cdots, X_n$ be our sample. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i, \quad s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$$

The kurtosis of X is

$$\kappa = \frac{E[(X-\mu)^4]}{\sigma^4}$$

For a normal RV the kurtosis is 3. So $\kappa - 3$ is sometimes called the excess kurtosis. Unfortunately sometimes the excess kurtosis is just called the kurtosis. In particular Owen does this.

(a) Find the variance of s^2 in terms of σ^2, κ and n. I think the answer is

$$\frac{1}{n}(\kappa\sigma^4 - \frac{n-3}{n-1}\sigma^4)$$

This is just a computation and rather tedious. You may want to skip this part and just do (b).

(b) Define

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2$$

This is just s^2 with the $1/(n-1)$ replaced by $1/n$. Expanding out the square we can also write this as

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2 - \overline{X}^2$$

Now let $Y = X^2$. The variance of X is $E[Y] - E[X]^2$ which is $f(E[X], E[Y])$ where $f(x, y) = y - x^2$. The estimator $\hat{\sigma^2}$ is $\overline{Y} - \overline{X}^2 = f(\overline{X}, \overline{Y})$. So we can use the delta method to find the variance of $\hat{\sigma^2}$. Carry out this computation and compare with the result in part (a).