Monte Carlo Methods - Spring 16 - Homework 5

1. Let X be the four points $\{(0,0),(1,0),(0,1),(1,1)\}$. We define a probability density on it by

$$f(0,0) = f(1,0) = 1/3, \quad f(0,1) = f(1,1) = 1/6$$

With the above ordering of the four points, we can think of this as a row vector $(1/3, 1/3, 1/6, 1/6)$. A transition kernel for this state space can be thought of as a 4 by 4 matrix.
(a) Consider the two stage Gibbs samplers. There are two of them since we can first update the second component and then the first component or we can first update the first component and then the second component. Find the transition matrices for these two algorithms. You should find that they are different, but the above row vector is the stationary distribution for both of them.
(b) Find the transition matrix for the randomized Gibbs sampler.
(c) Find the transition matrix for the "wrong" two-stage Gibbs sampler discussed in class and the notes. Is the above row vector the stationary distribution for it?

2. Consider the randomized Gibbs sampler for a distribution $f(x_1, x_2, \cdots, x_d)$ which is absolutely continuous with respect to d-dimensional Lebesgue measure. State precisely and prove a theorem that says that if the support of f is "thickly" connected, then the Markov chain is irreducible. Hint: mimic what we did to prove irreducibility of the Metropolis-Hastings algorithm.

3. Proposition 3 in chapter 8 of the notes says that for the independence sampler if $\pi(x) \leq Cg(x)$ where $g(x)$ is the proposal distribution, then the chain converges geometrically fast in the total variation norm. In class we proved this theorem for the case that the distribution is continuous with the additional assumption that the initial distribution was continuous. Do one (or both!) of the following.
(a) Prove that if we start the chain in a single state, then the chain still converges geometrically fast in the total variation norm.
(b) Prove the theorem when $\pi(x)$ is a discrete distribution, i.e., the state space is discrete.

4. Consider the last example in section 8.4 on the Gibbs sampler. Suppose that the $p_i(x)$ are all the same and are Poisson with parameter λ. So

$$p_i(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad for \quad x = 0, 1, 2, \cdots$$

Implement the Gibbs sampler discussed in class and use it to compute the expected value of X_1 when $d = 2$ and $m = 10$. Note: this can be computed analytically to check your simulation result.