Renormalization group maps for Ising models and tensor networks

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Webinar “Analysis, Quantum Fields, and Probability”
December 9, 2021

Outline

Joint work with Slava Rychkov

- Wilson-Kadanoff RG (real-space RG)
- Tensor networks
- Simple RG map for tensor network
- High temperature fixed point - stable?
- Problem : eigenvalue=1 - CDL problem
- Better RG - disentangler
- Stability of high temp fixed point
- Outlook

arXiv:2107.11464

Ask questions.

Wilson-Kadanoff RG (real space RG)

Ising type models: spins take on only values ± 1.
Nearest neighbor interaction

$$H(\sigma) = -\beta \sum_{<i,j>} \sigma_i \sigma_j$$

More general interaction

$$H(\sigma) = \sum_{Y} d(Y)\, \sigma(Y), \qquad \sigma(Y) = \prod_{i \in Y} \sigma_i$$

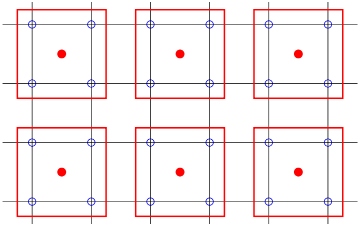
where the sum is over all finite subsets including the empty set.

Note that β has been absorbed into the Hamiltonian.

Blocking

Lattice divided into blocks; each block assigned a block spin variable.

Block spins also take on only the values ± 1.



Wilson Kadanoff RG

Original spins: σ Block spins: $\overline{\sigma}$

RG Kernel: $T(\overline{\sigma}, \sigma)$, e.g., majority rule

Satisfies

$$\sum_{\overline{\sigma}} T(\overline{\sigma}, \sigma) = 1, \quad \forall \sigma$$

for all original spin configurations σ.

Renormalized Hamiltonian $\bar{H}(\overline{\sigma})$ is formally defined by

$$e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} T(\overline{\sigma}, \sigma)\, e^{-H(\sigma)}$$

Note: β has been absorbed into the Hamiltonians.

Key point: only makes sense in finite volume.

RG maps preserves Z

$$\sum_{\overline{\sigma}} T(\overline{\sigma},\sigma) = 1 \quad \forall\sigma, \qquad\qquad e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} T(\overline{\sigma},\sigma)\, e^{-H(\sigma)}$$

$$\sum_{\overline{\sigma}} e^{-\bar{H}(\overline{\sigma})} = \sum_{\sigma} e^{-H(\sigma)}$$

So free energy of the original model can be recovered from the renormalized Hamiltonian.

Study the critical behavior of the system by studying iterations of the renormalization group map:

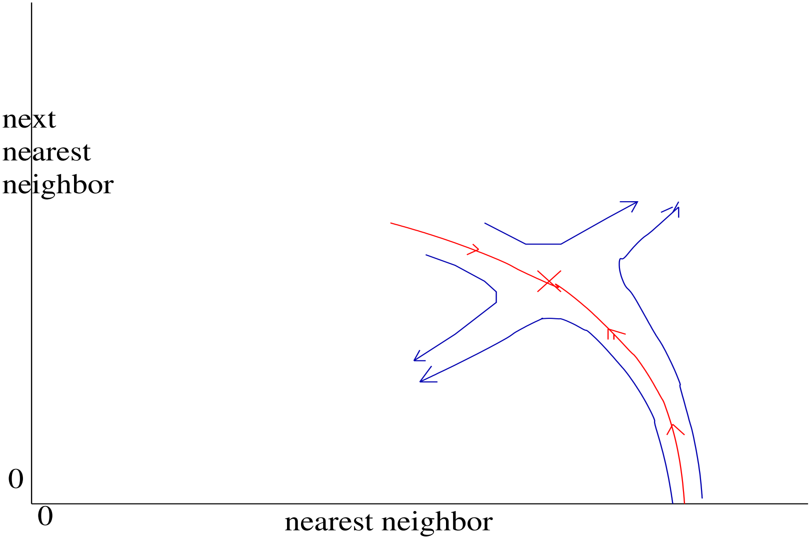
$$\mathcal{R}(H) = \bar{H}$$

Remember: $\mathcal{R}$ is not even defined from a rigorous point of view.

RG flow

Has a fixed point with stable manifold of co-dim 2

Eigenvalues of linearization $> 1 \implies$ critical exponents



Rigorous results

Existence of map at high temp or large magnetic field
Griffiths and Pearce; Israel; Kashapov; Yin

Non-existence of map at low temp for various kernels
Griffiths and Pearce; Israel; van Enter, Fernández and Sokal

Non-existence of map near critical temp for some kernels

Essentially no results even for first iteration of the map near critical surface.

Goal: Not to determine for each T whether it works or not.
Show there is one T that works.

Numerical studies

Wilson (Rev. Mod. Phys. 1975) - 217 terms in H !
"A number of details are omitted."

Lots of Monte Carlo studies using Wilson-Kadanoff RG

Swendsen: compute the linearization of the RG map from correlation functions. Avoids computing $\mathcal{R}$ itself.

Brandt,Ron,Swendsen Saw significant dependence of $\bar{H}$ on truncation method.

"Even though the individual multispin interactions usually have smaller coupling constants than two-spin interactions, the fact that they are very numerous can lead to multispin interactions dominating the effects of two-spin interactions."

Lattice gas variables

RG calculations usually done using the spin variables $\sigma_i = \pm 1$.

lattice gas variables: $n_i = (1 - \sigma_i)/2$ which take on the values $0, 1$.

In lattice gas variables

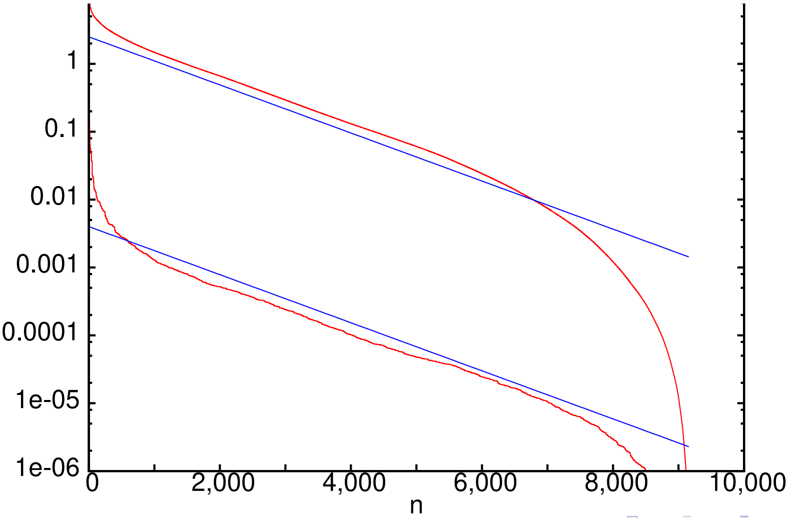
$$\bar{H}(\bar{n}) = \sum_Y c(Y)\,\bar{n}(Y), \qquad \bar{n}(Y) = \prod_{i \in Y} \bar{n}_i$$

Y summed over all finite subsets of block spins
Take H to be n.n. critical Ising
You can compute the $c(Y)$ very accurately.
Compute them for about $10,000$ $Y's$.
Order by decreasing $|c(Y)|$ and plot.
arXiv:0905.2601

Decay of lattice gas coefs

Bottom curve: $|c(Y_n)|$ vs. n. Top curve: $\sum_{i=n}^{N} |c(Y_n)|$ vs. n.

Two lines : $c2^{-n/850}$



Open problems

1. Prove there is a Banach space of Hamiltonians and a rigorously defined RG map on it which has a non-trival fixed point with a stable manifold of co-dimension two.

2. Develop a systematic numerical approach to compute the RG map.

Tensor network

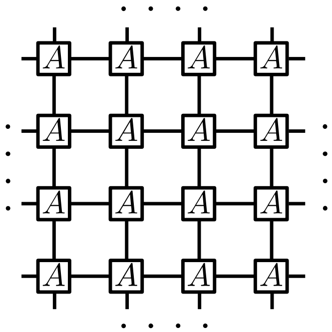
Let H be a real Hilbert space (finite or infinite dimensional)
A tensor (of order 4) is a map

$$A : H \times H \times H \times H \to \mathbb{R}$$

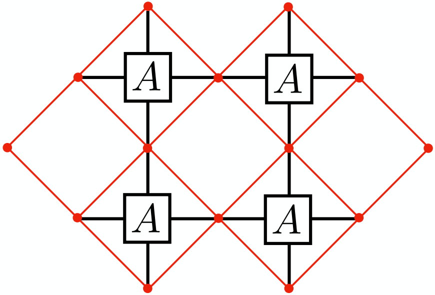
which is linear in each argument. Let e_i be o.n. basis for H.

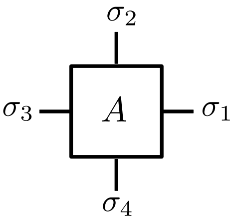
$$A_{ijkl} = A(e_i, e_j, e_k, e_l)$$

Tensor network is formed by contracting copies of A:



Ising model as tensor network - cond matter

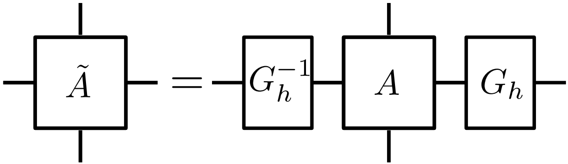




$$= e^{\beta(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)}$$

Gauge transformations

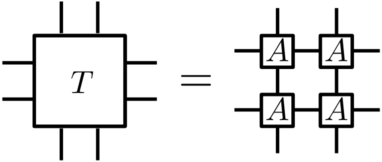
Let G_h be invertible 2-leg tensor (matrix). Define $\tilde{A}$ by

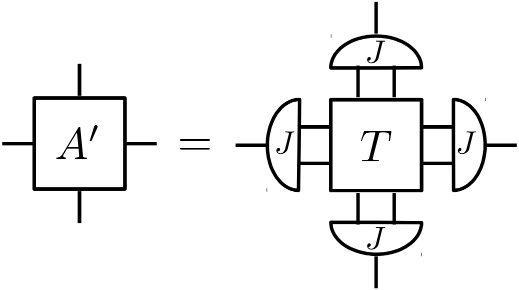


Contraction of $\tilde{A}$ network is same as contraction of A network.

Simplest RG for tensor network

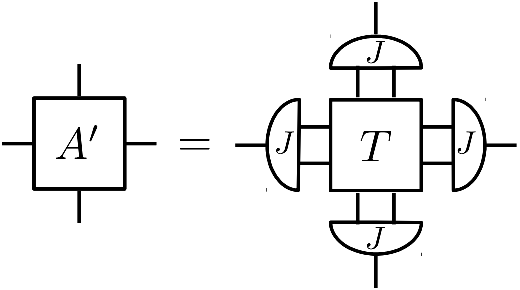
Levin,Nave (2007)





J is isometry of $H \otimes H$ onto H.

Simplest RG for tensor network



J is isometry of $H \otimes H$ onto H. Many such isometries.
This freedom is equivalent to a gauge transformation.

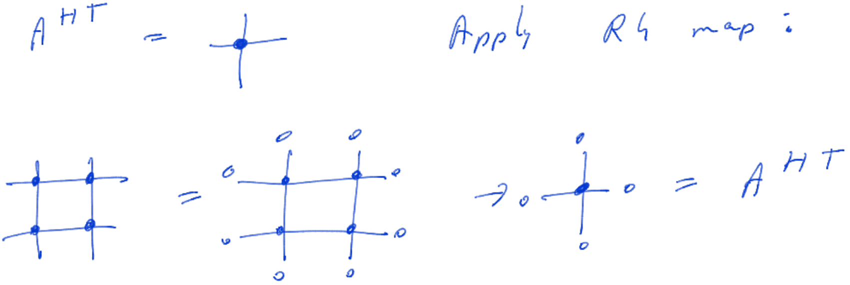
Wilson-Kadanoff vs tensor network RG

- *Growth of number of variables*
  WK RG: Spins only have two values but Hamiltonian becomes non-local with many multi-body terms
  TN RG: Tensor stays local, but leg dimension grows
- *Computability of RG map*
  WK RG: no explict way to compute it - ∞ volume limit
  TN RG: Explicitly computable, but disentanglers complicate it

High temperature fixed point

Let A^{HT} be tensor with one nonzero component $A^{HT}_{0000} = 1$.

Assume $J(e_0 \otimes e_0) = e_0$. Then A^{HT} is a fixed point of RG.



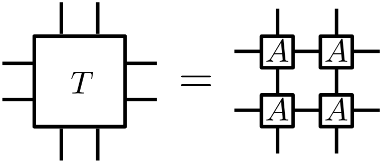
Is it stable?

Norms

Use Hilbert-Schmidt (Frobenius) norm:

$$||A||^2 = \sum_{ijkl} A_{ijkl}^2$$

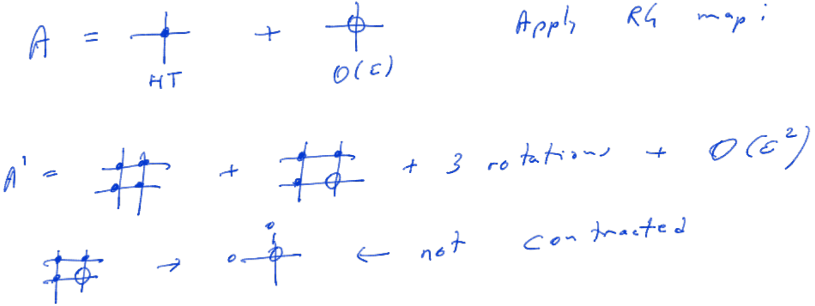
If A, B are tensors of any order and C is formed by contracting some indices of A with some indices of B, then by Cauchy-Schwaz inequality $||C|| \leq ||A|| ||B||$.



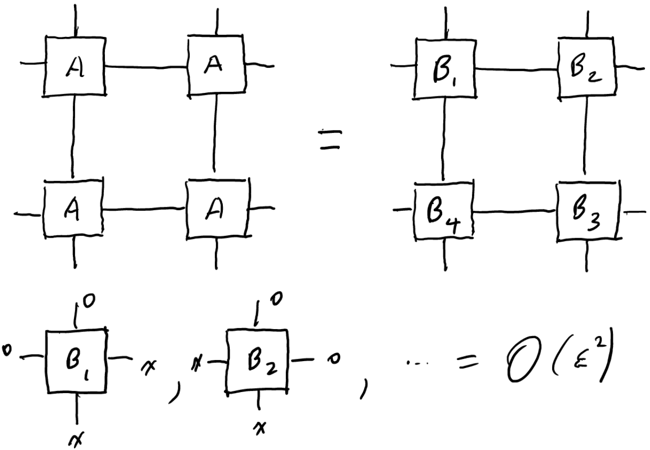
$$||T|| \leq ||A||^4$$

CDL Problem (Corner double line)

Now perturb A^{HT}: $A = A^{HT} + \delta A$, $||\delta A|| = O(\epsilon)$.
Compute A' to first order:

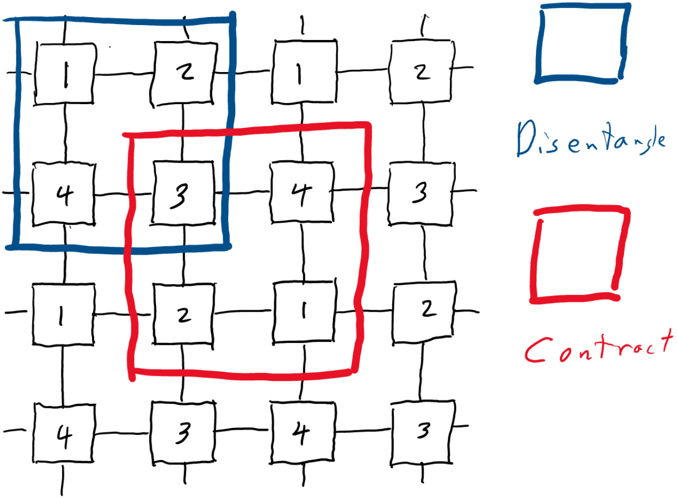


Disentanglers



0: leg fixed to 0 index, x: leg can only be nonzero index
subtensor

Disentanglers - cont



Stability of high temp fixed point

Theorem:

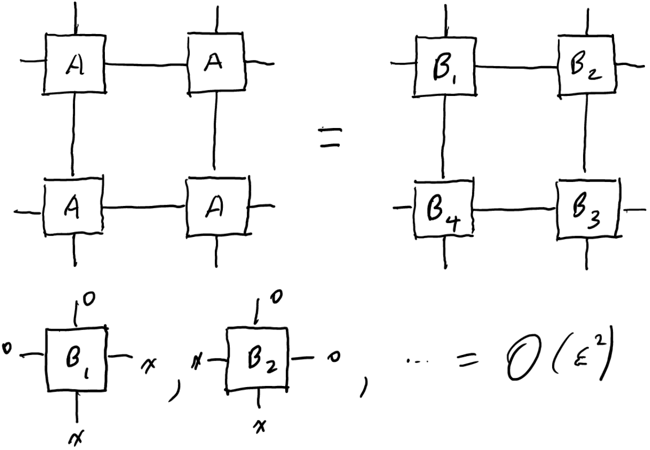
There is a tensor RG map such that if $A = A^{HT} + \delta A$ with $||\delta A||$ small, then the image has the form $A' = A^{HT} + \delta A'$ with

$$||\delta A'|| \leq C||\delta A||^{3/2}$$

(The tensor A is normalized so that $A_{0000} = 1$ and the RG map includes a normalization step so that $A'_{0000} = 1$.)

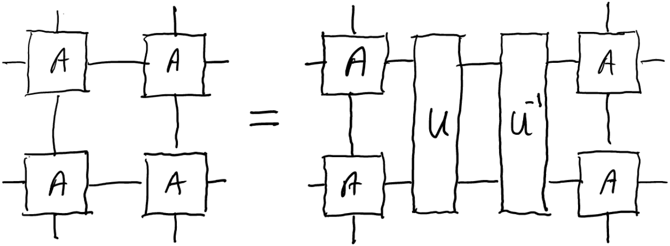
Sketch of the proof - 1

Recall the disentangler :

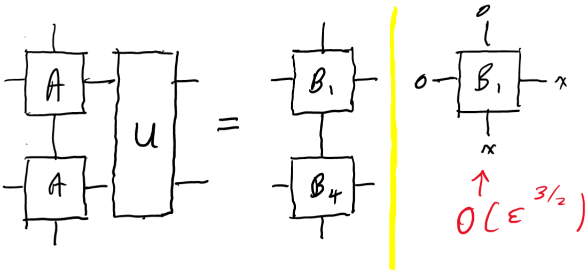


This reduces the proof to proving the existence of the disentangler.
NB: We will cheat a bit in the following - more on this later

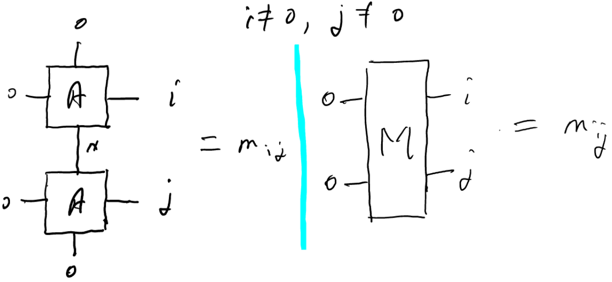
Sketch of the proof - 2



Sketch of the proof -3



Sketch of the proof -4



$$M = \sum_{i\neq 0, j\neq 0} m_{ij} \left|\begin{smallmatrix}0\\0\end{smallmatrix}\right\rangle \left\langle\begin{smallmatrix}i\\j\end{smallmatrix}\right|$$

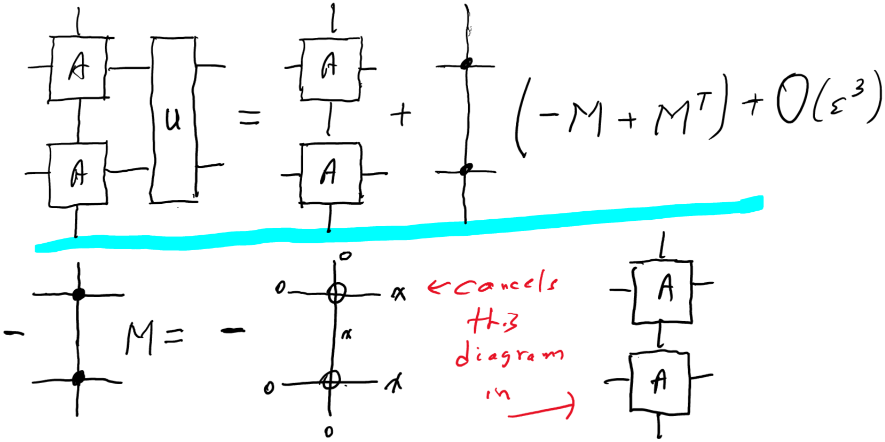
$$U = \exp(-M + M^T)$$

Note $||M|| = O(\epsilon^2)$.

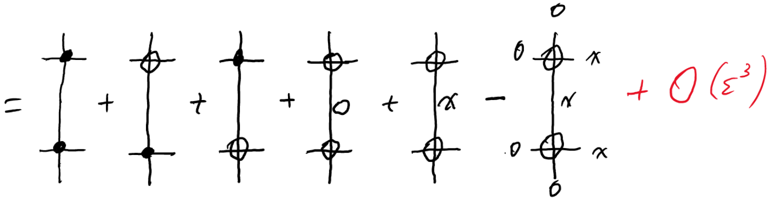
$$U = I - M + M^T + O(\epsilon^4)$$

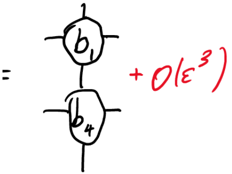
Sketch of the proof -5

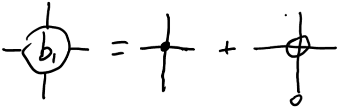
Compute to order $O(\epsilon^2)$



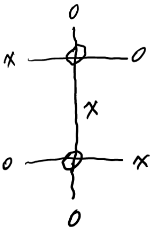
Sketch of the proof - 6







Sketch of the proof - the cheat



not cancelled

not $O(\varepsilon^3)$

Outlook

Presented a modest first step in rigorous study of tensor RG maps without truncation.

Holy grail : prove there is a RG map for tensor networks with a non-trivial fixed point.

There are many tensor RG maps that have been studied numerically. Which one is best for above?