Efficient SLE algorithms and numerical pitfalls of the method

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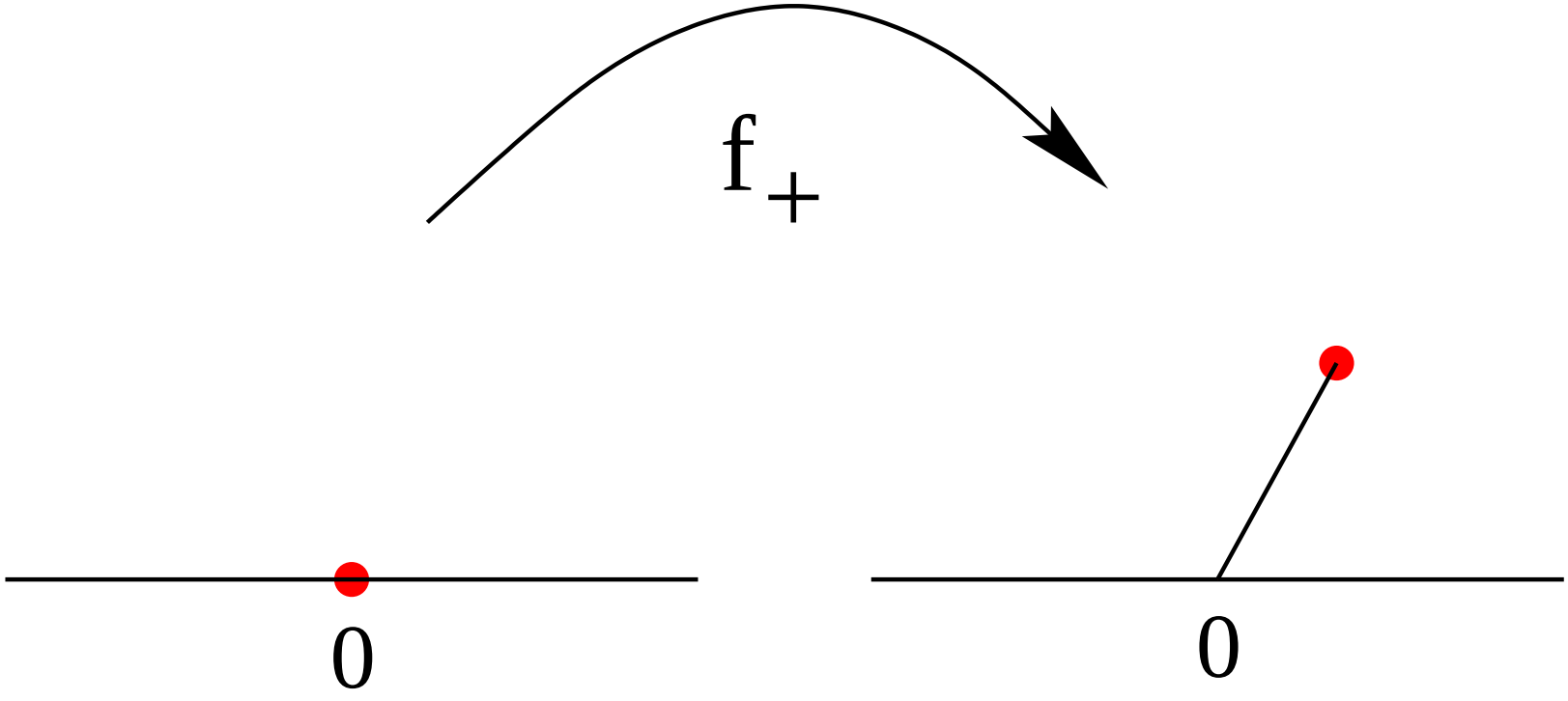
Outline

Theme : How reliable is testing for SLE by computing the Loewner driving function ?

- SLE as iterated random conformal maps
- Computing the driving process of a curve
- Testing for SLE
  - Self-avoiding walk (SAW)
  - Distorted models
  - Gradient-gradient percolation
- Conclusions and caveats

SLE in two minutes

f_+ =conformal map of half plane to half plane minus slit

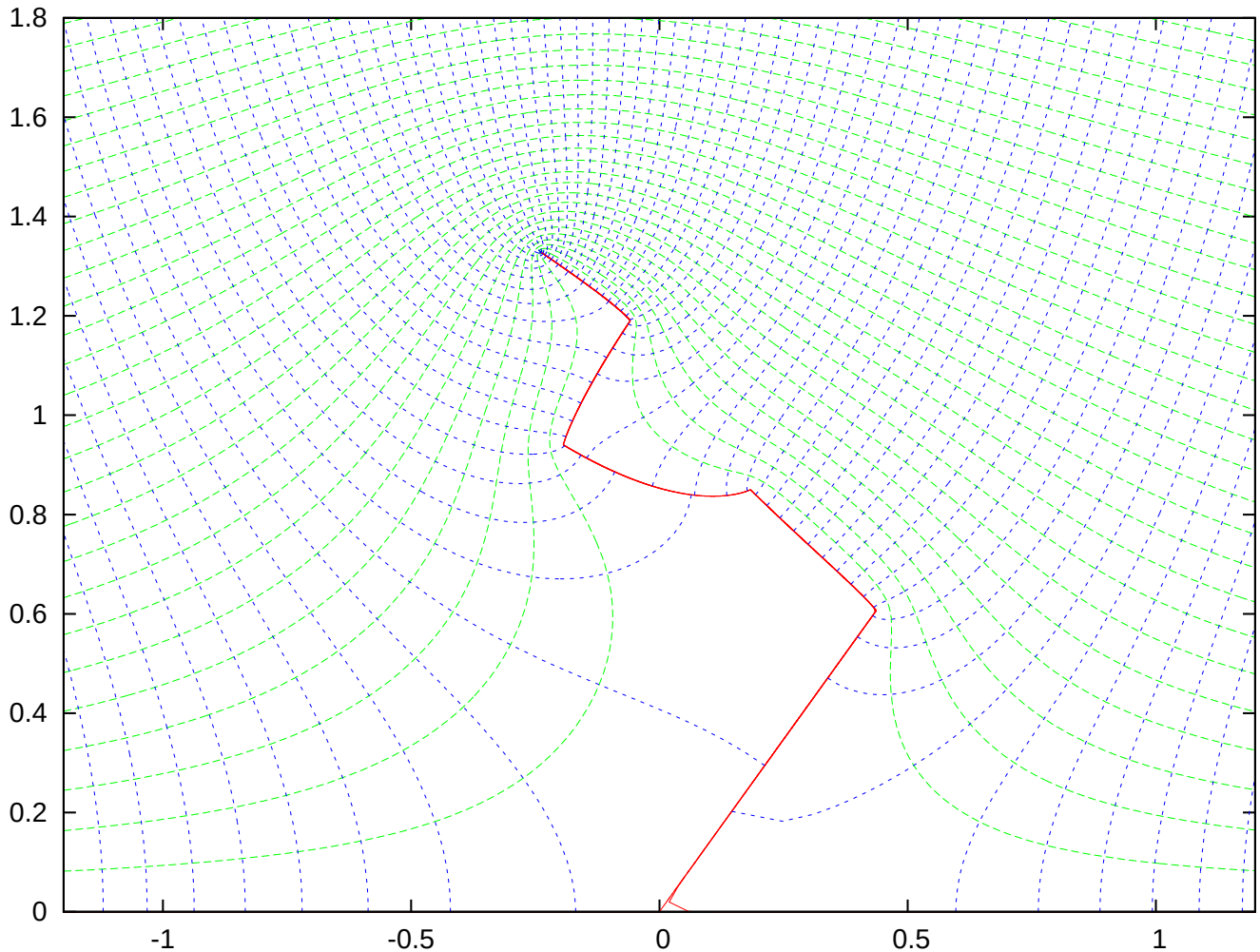


$$f_+(0) = tip, \qquad f_+(z) = z + \Delta x - \frac{2\Delta t}{z} + O(\frac{1}{z^2}), \quad z \to \infty$$

f_- is reflection of f_+ to insert slit to left

SLE in two minutes

Look at $f_+ \circ f_- \circ f_- \circ f_+ \circ f_-$

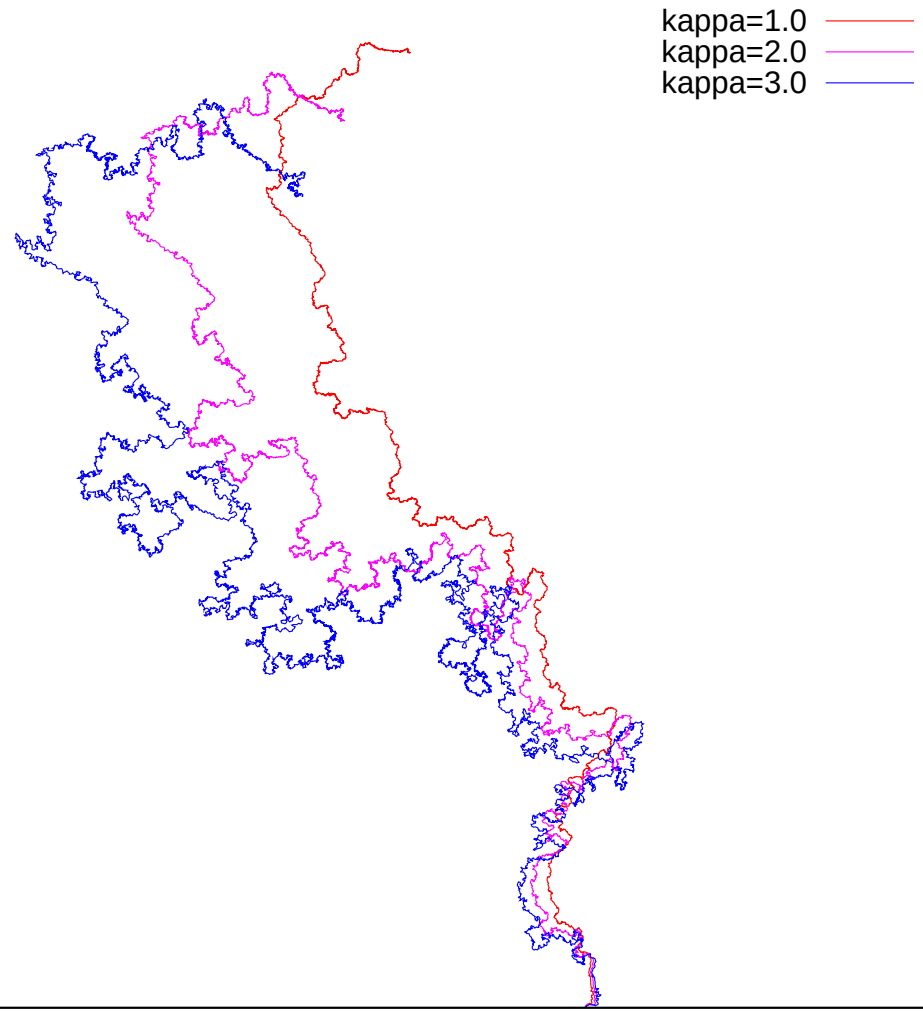


SLE in two minutes

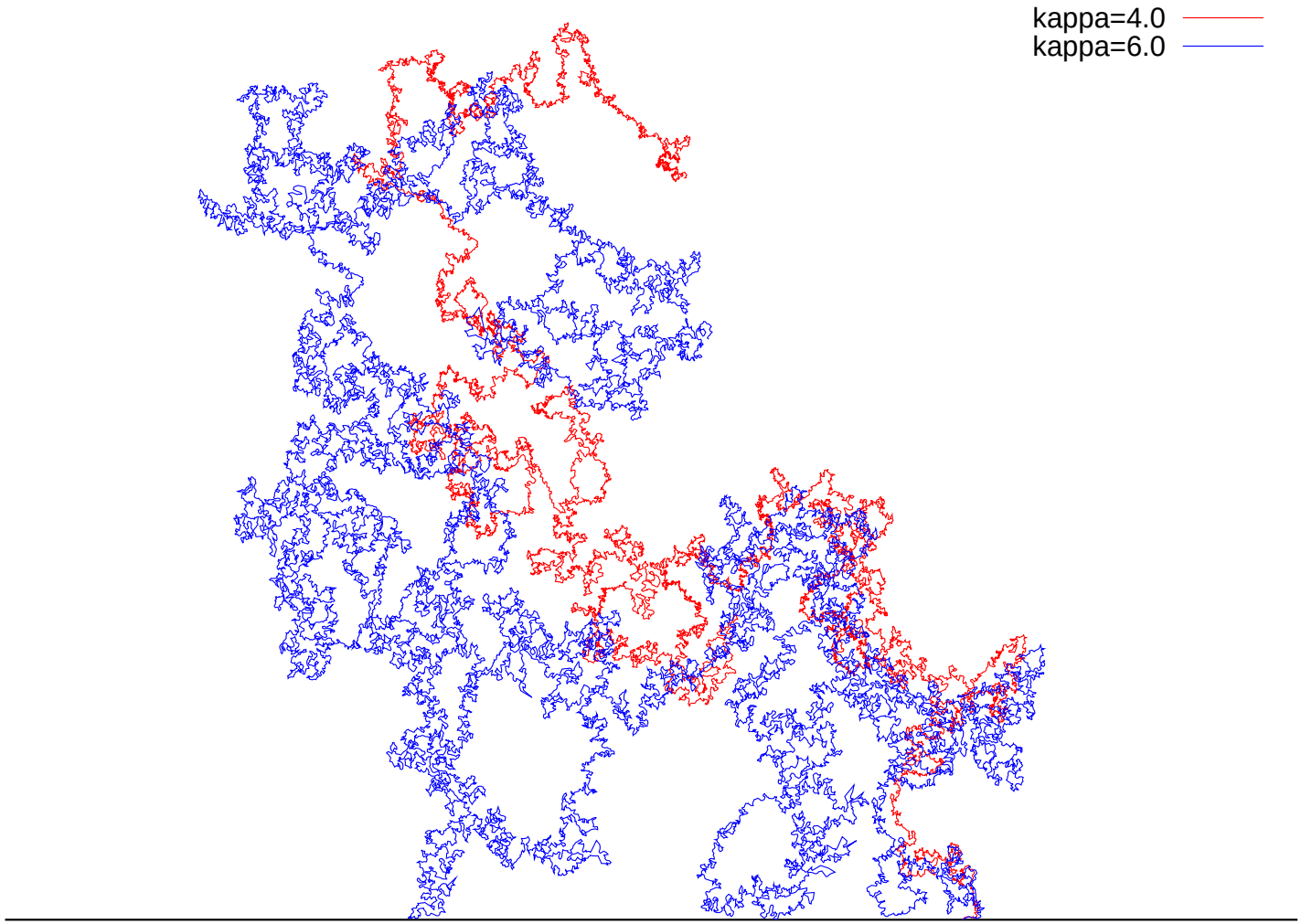
Look at $f_{s_n} \circ f_{s_{n-1}} \circ \cdots \circ f_{s_2} \circ f_{s_1}$, s_k are independent, $\pm$ with prob $1/2$.

Let length of slit go to zero. $\Delta x, \Delta t \to 0$

$\kappa = (\Delta x)^2/\Delta t$ is independent of length, depends on angle.



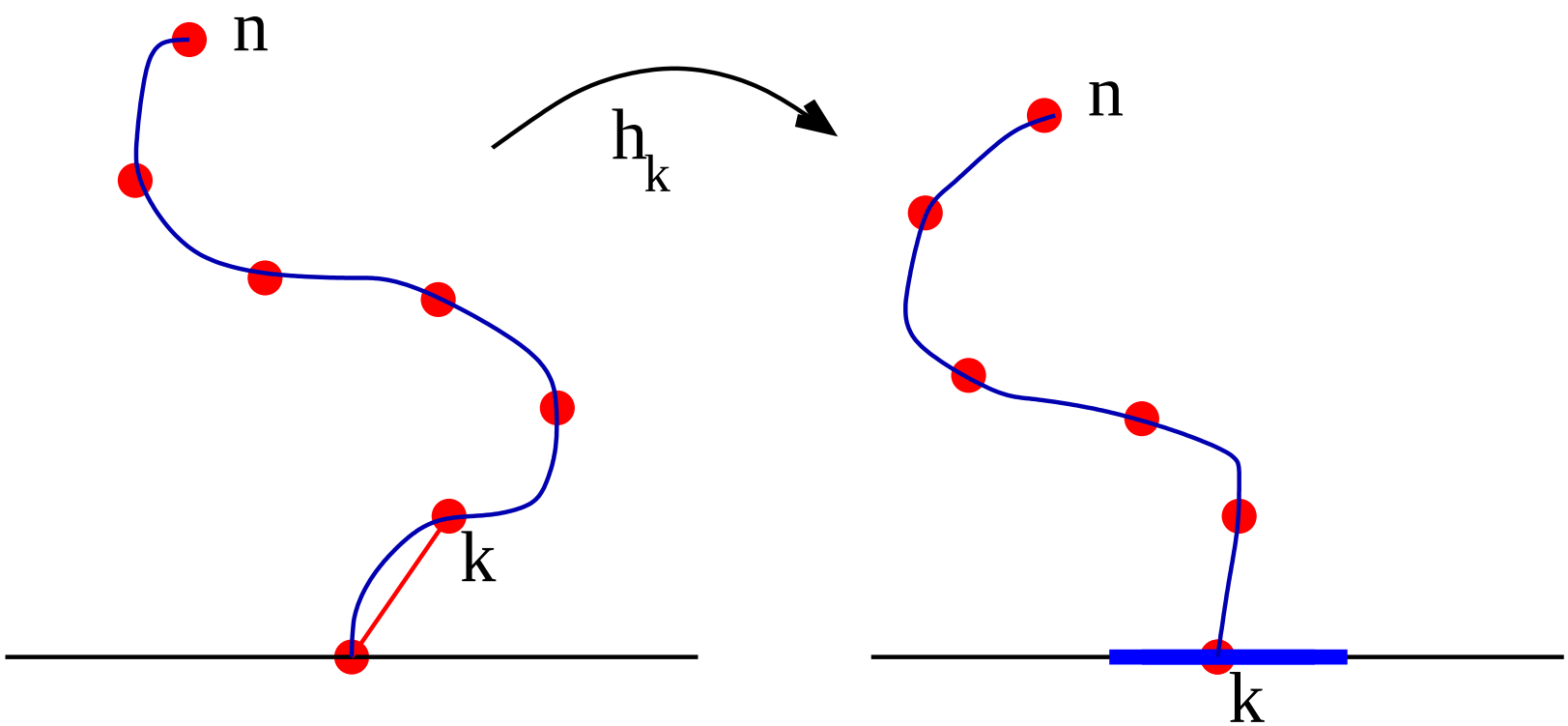
SLE in two minutes



Computing the driving process of a curve

Start with points on a curve in upper half plane (possibly random).
"Unzip" it one point at a time.

At kth stage h_k is conformal map that



$$h_k(tip) = 0, \quad h_k(z) = z - \Delta x_k + \frac{2\Delta t_k}{z} + O(\frac{1}{z^2})$$

Computing the driving process of a curve

The function $U(t)$ given by

$$U(\sum_{i=1}^{k} \Delta t_i) = \sum_{i=1}^{k} \Delta x_i$$

is called the driving function.

For our iterated random conformal maps it is a simple random walk, converges to Brownian motion with $var(U_t) = \kappa t$.

Caution: the parametrization t of the curve is unnatural

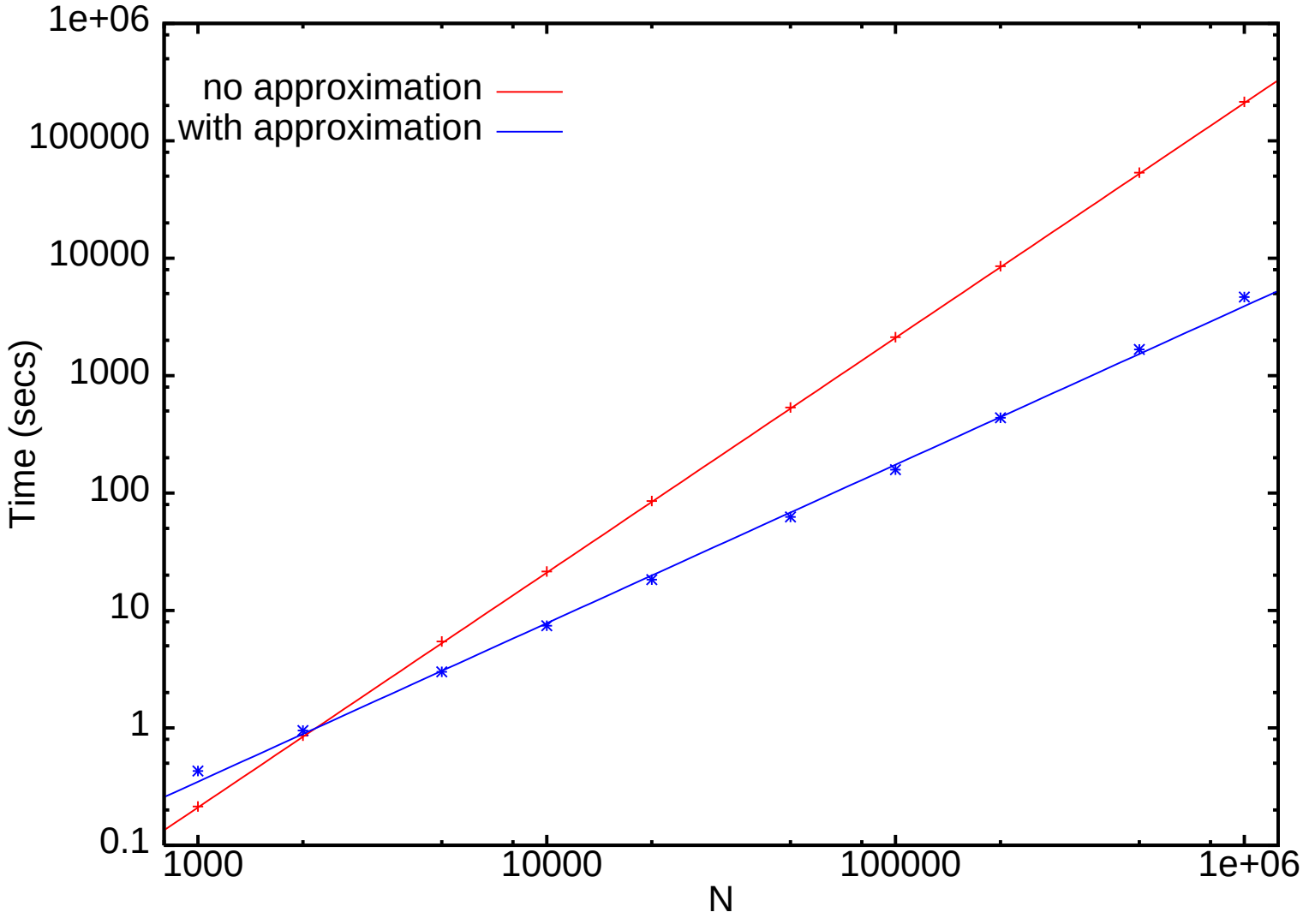
Naive implementation : with N points on curve

Each step must compute image of $O(N)$ points $\Rightarrow$ $O(N^2)$ time.

Computing the driving process of a curve

You can speed this up with some tricks involving Laurent series. Unzip a SAW. Plot time (in seconds) vs. N, number of step.

Lines have slope 2 and 1.35.



Computing the driving process of a curve - details

Single w_{k+1} takes $O(k)$ time. All the w_{k+1} takes $O(N^2)$.

$$w_{k+1} = h_k \circ h_{k-1} \circ \cdots \circ h_1(z_{k+1})$$

Group functions we are composing into blocks.

$$H_j = h_{jb} \circ h_{jb-1} \circ \cdots \circ h_{(j-1)b+2} \circ h_{(j-1)b+1}$$

With $k = mb + r$ with $0 \leq r < b$,

$$w_{k+1} = h_{mb+r} \circ h_{mb+r-1} \circ \cdots \circ h_{mb+1} \circ H_m \circ H_{m-1} \circ \cdots \circ H_1(z_{k+1})$$

Approximate composition within a block using a series.

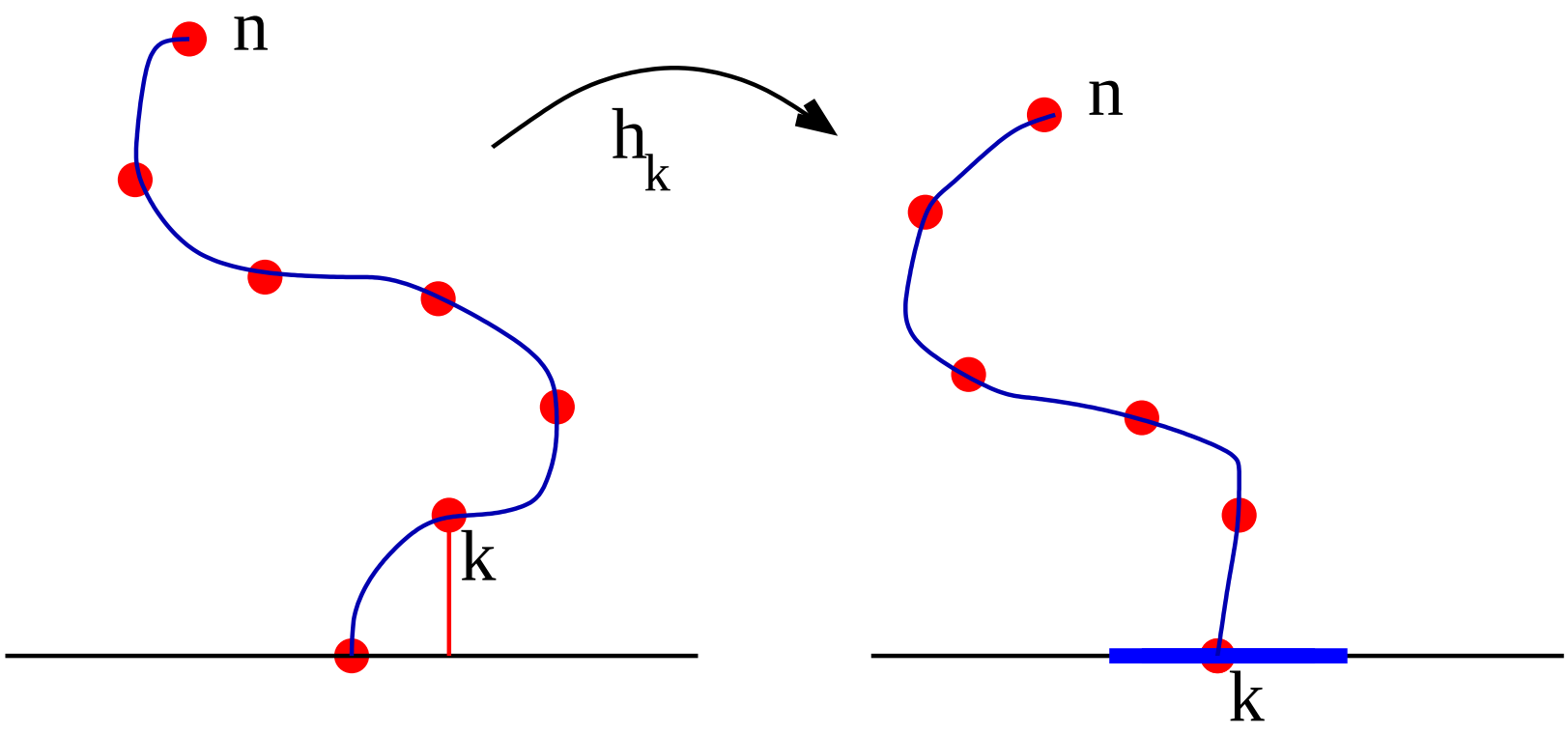
Compute this approximation to H_j just once.

Don't alway use the approximation to H_j.

Computing the driving process of a curve

Other h_k : do not have to use tilted slit.

Vertical slit much easier to compute:



Testing for SLE

SLE corresponds to driving process being Brownian motion.

Given a collection of random curves we compute their driving processes and test if they are a sample of Brownian motion.

Brownian motion:

1. U_t is normal, mean 0, variance κt.
2. $0 < t_1 < t_2 < \cdots t_n$: increments $U_{t_n} - U_{t_{n-1}}, \cdots, U_{t_2} - U_{t_1}, U_{t_1}$

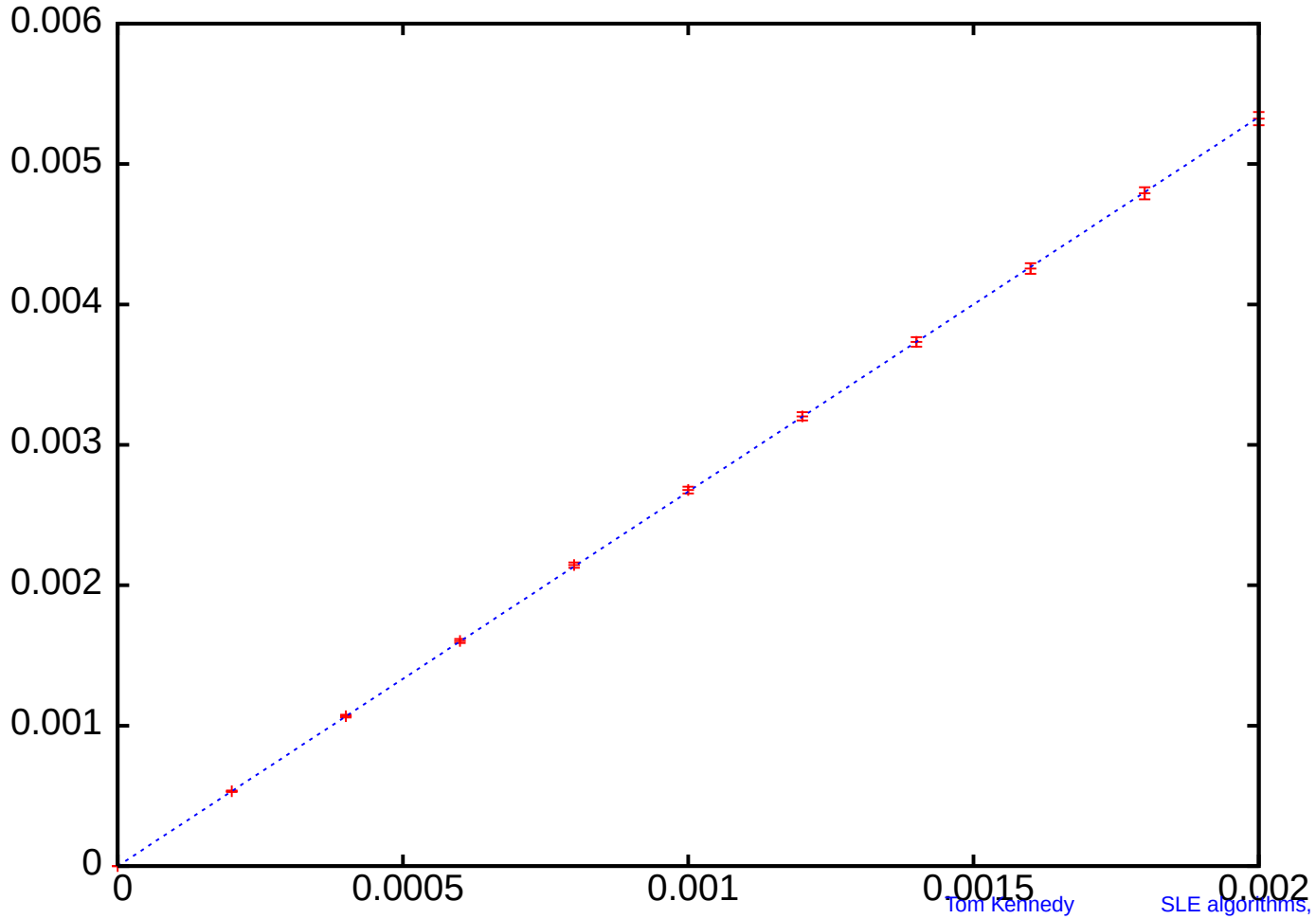
   They are independent, normal.

Testing for SLE - SAW

SAW with 200,000 steps; only compute driving function up to a fixed capacity T. Average number of steps in "unzipped" portion is 9350.

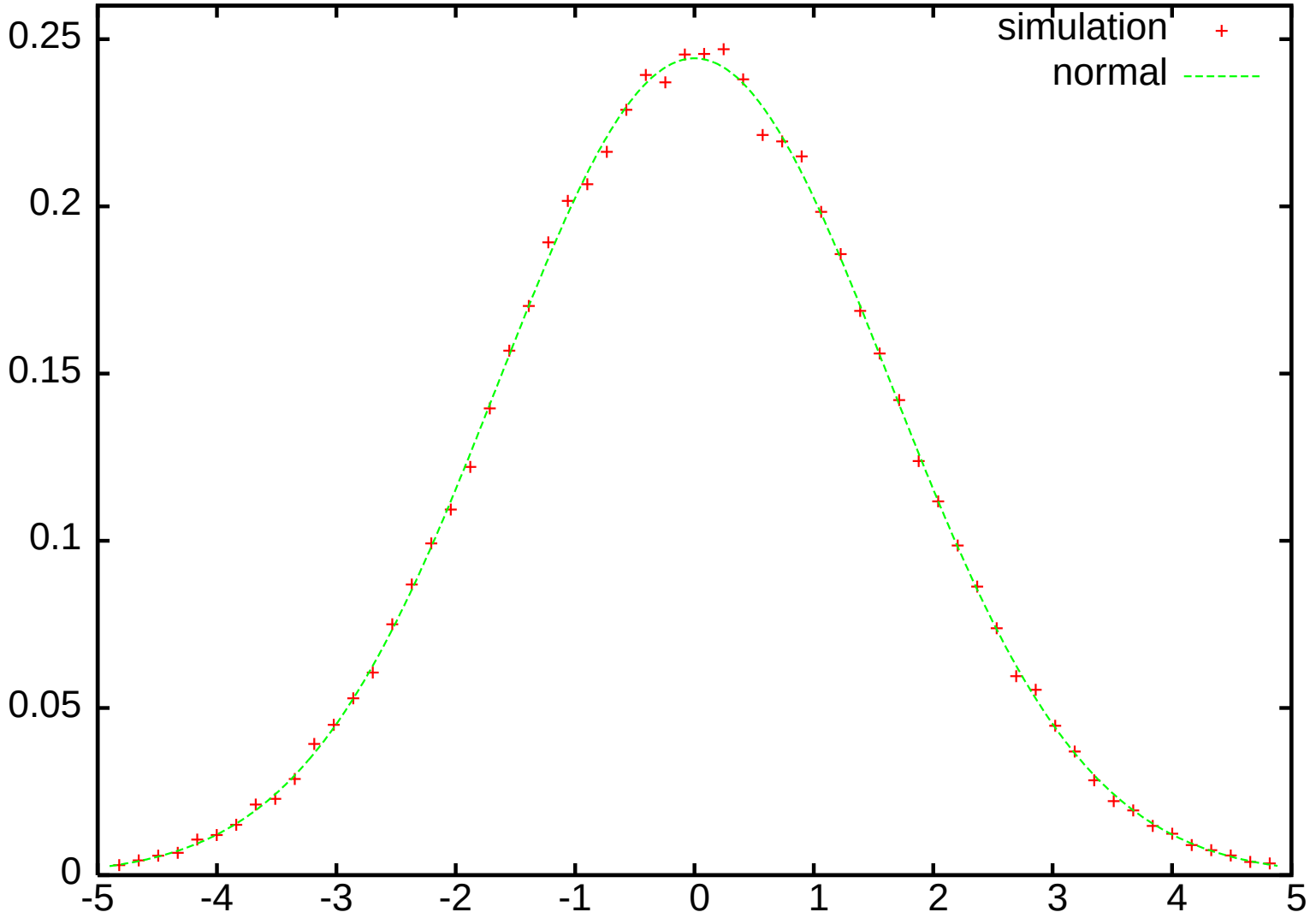
100,000 samples, i.e., SAW's

Plot variance of driving process, compare slope to $8/3$: 2.6686 ± 0.0132.



Testing for SLE - SAW

Distribution of driving function at capacity T.



Testing for SLE - distorted models

Take a model which converges to SLE and distort it :

Distort curves: $(x, y) \to (x, \lambda y)$

Easy to see this is not another SLE.

No simple relation between the driving function for γ and distorted γ.

Three models : loop-erased random walk (LERW), SAW, percolation

We compute 100,000 samples of SAW and distort them,
$\lambda = 0.9, 0.95, 1., 1.05, 1.1$.

Then we compute the driving function of the distorted curve up to a fixed capacity that corresponds to about 10,000 steps.

Testing for SLE - distorted models

Result: collection of samples of the driving process of the distorted model.

Do various statistical tests to see if this process is a Brownian motion.

Invariance under $(x, y) \to (-x, y)$ implies $E[U_t] = 0$.

First plot variance $E[U_t^2]$ vs. t.

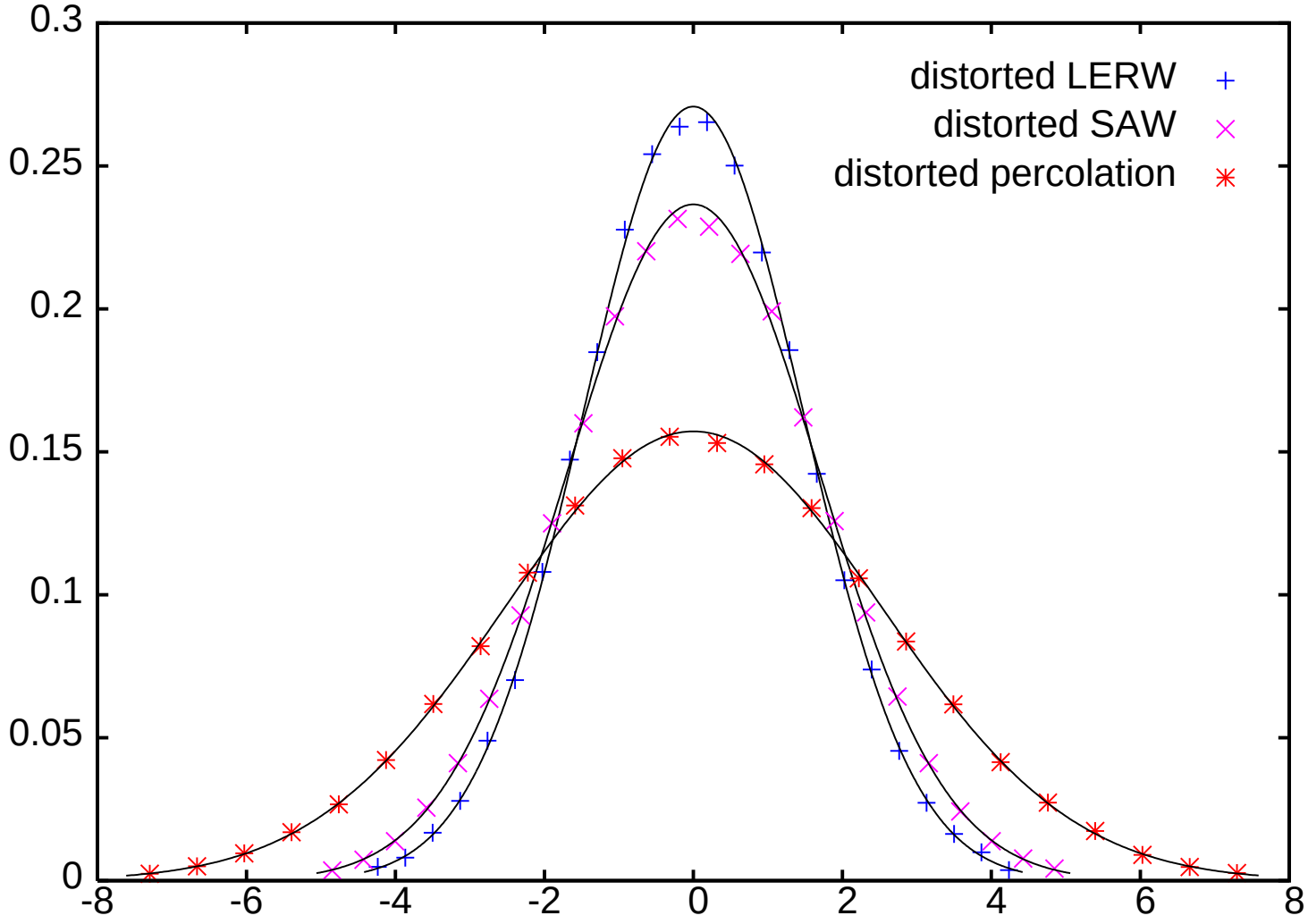
Scaling $\Rightarrow E[U_t^2] = \kappa t$, even if scaling limit is not an SLE.

Model	λ	κ
LERW	0.95	2.1709 ± 0.0094
LERW	1.00	2.0008 ± 0.0093
SAW	0.95	2.8414 ± 0.0108
SAW	1.00	2.6686 ± 0.0132
percolation	0.95	6.4422 ± 0.0311
percolation	1.00	6.0404 ± 0.0265

Testing for SLE - distorted models

Points are a histogram for the density of $U_T/\sqrt{T}$ for the LERW, SAW and percolation with distortion $\lambda = 0.95$.

Curves are normal densities, variance from a least square fit of ....



Testing for SLE - distorted models

First statistical test: fix t, is distribution of U_t normal?

Kolmogorov-Smirnov test: statistic D (function of random sample)

Under the hypothesis that U_t is normal with given variance, distribution of D is known.

Large $D \Rightarrow$ not normal

Use "p-values" to quantify "large D".

Let D_0 be the value of D we get, then $p = P(D \geq D_0)$.

We perform this Kolmogorov-Smirnov test for two values of the time, T and $T/2$.

Testing for SLE - distorted models

λ	samples	$T/2$	T
0.80	5,000	0.324723	0.118259
	10,000	0.026838	0.056587
	20,000	0.074464	0.010112
	50,000	0.093682	0.004555
	100,000	0.004130	0.000181
0.95	5,000	0.408348	0.350261
	10,000	0.136263	0.888704
	20,000	0.805490	0.663008
	50,000	0.483299	0.323554
	100,000	0.313017	0.158268
1.00	5,000	0.456948	0.371554
	10,000	0.557806	0.267715
	20,000	0.933089	0.552806
	50,000	0.956776	0.852597
	100,000	0.501460	0.870474

Testing for SLE - distorted models

Tests of Brownian motion:

Let $0 < t_1 < t_2 < \cdots < t_n = T$, $X_j = U_{t_j} - U_{t_{j-1}}$.

For Brownian motion, X_j are independent; each is normal with mean zero and variance $\kappa(t_j - t_{j-1})$.

Test this joint distribution with a χ^2 goodness-of-fit test.

Divide the possible values of $(X_1, X_2, \cdots, X_n)$ into m cells,

Count number of samples in each cell, O_j.

Brownian motion $\Rightarrow$ expected number is E_j.

$$\chi^2 = \sum_{j=1}^{m} \frac{(O_j - E_j)^2}{E_j}$$

Testing for SLE - distorted models

Three choices of cells

a. $n = 10$, use only signs of the X_j. So $2^{10} = 1024$ cells.

b. $n = 5$, use only signs of the X_j. So $2^5 = 32$ cells.

c. $n = 2$. Look at quartile of X_1 and X_2. So 16 cells.

First two statistics do not involve κ.

First two statistics do not test normality of U_t, only that it is symmetric.

Testing for SLE - distorted models

λ	samples	χ^2_a	χ^2_b	χ^2_c
0.80	5,000	0.001274	0.000000	0.110155
	10,000	0.000000	0.000000	0.000001
	20,000	0.000000	0.000000	0.000000
	50,000	0.000000	0.000000	0.000000
	100,000	0.000000	0.000000	0.000000
0.95	5,000	0.027830	0.037820	0.883070
	10,000	0.775130	0.704539	0.931479
	20,000	0.682962	0.797951	0.104660
	50,000	0.162307	0.030086	0.959757
	100,000	0.000018	0.009703	0.534966
1.00	5,000	0.618423	0.126116	0.424437
	10,000	0.200096	0.145614	0.609878
	20,000	0.793825	0.253454	0.370379
	50,000	0.685205	0.275748	0.161052
	100,000	0.028353	0.079678	0.579479

Testing for SLE - gradient-gradient percolation

We consider another model that is not SLE that we refer to as gradient-gradient percolation.

For off critical percolation on a unit lattice, the length scale is

$$L = (p - p_c)^{-4/3}$$

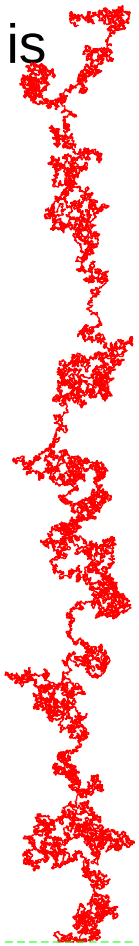
Gradient percolation we can take

$$p = p_c + cx$$

The choice of c sets a length scale.

Exploration process will live in a vertical strip.

Driving process will be stationary for large t.



Testing for SLE - gradient-gradient percolation

$$L = (p - p_c)^{-4/3}$$

Gradient-gradient percolation:

$$p = p_c + cy^{-7/4}x$$

At height y, "width" of the path is given by setting the length scale for p equal to x:

$$x = [y^{-7/4}x]^{-4/3} = y^{7/3}x^{-4/3}$$

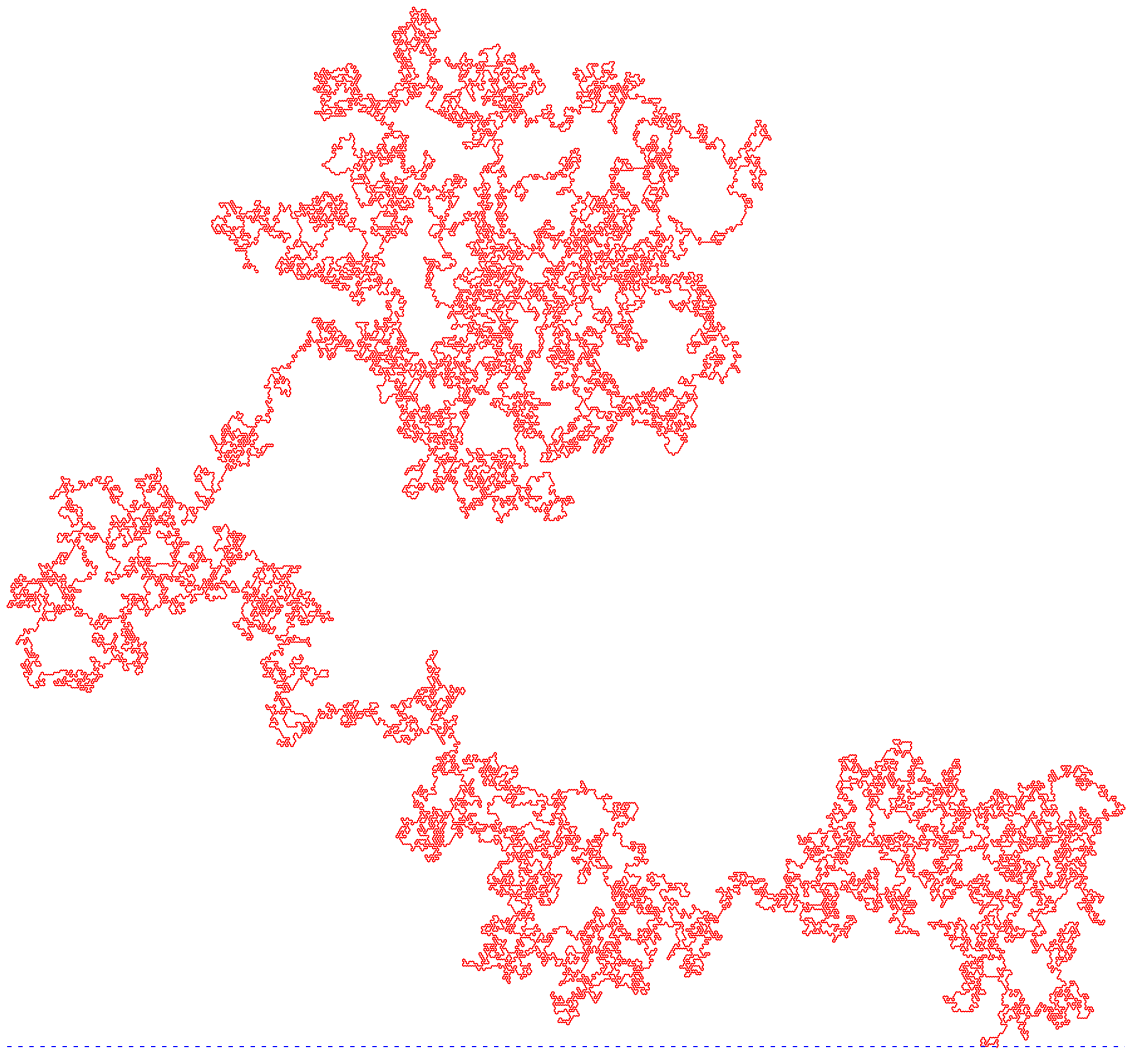
So $y = x$, i.e., the horizontal width is of the same order as the height.

Scaling limit should be invariant under dilations

Driving function variance linear in t.

Testing for SLE - gradient-gradient percolation

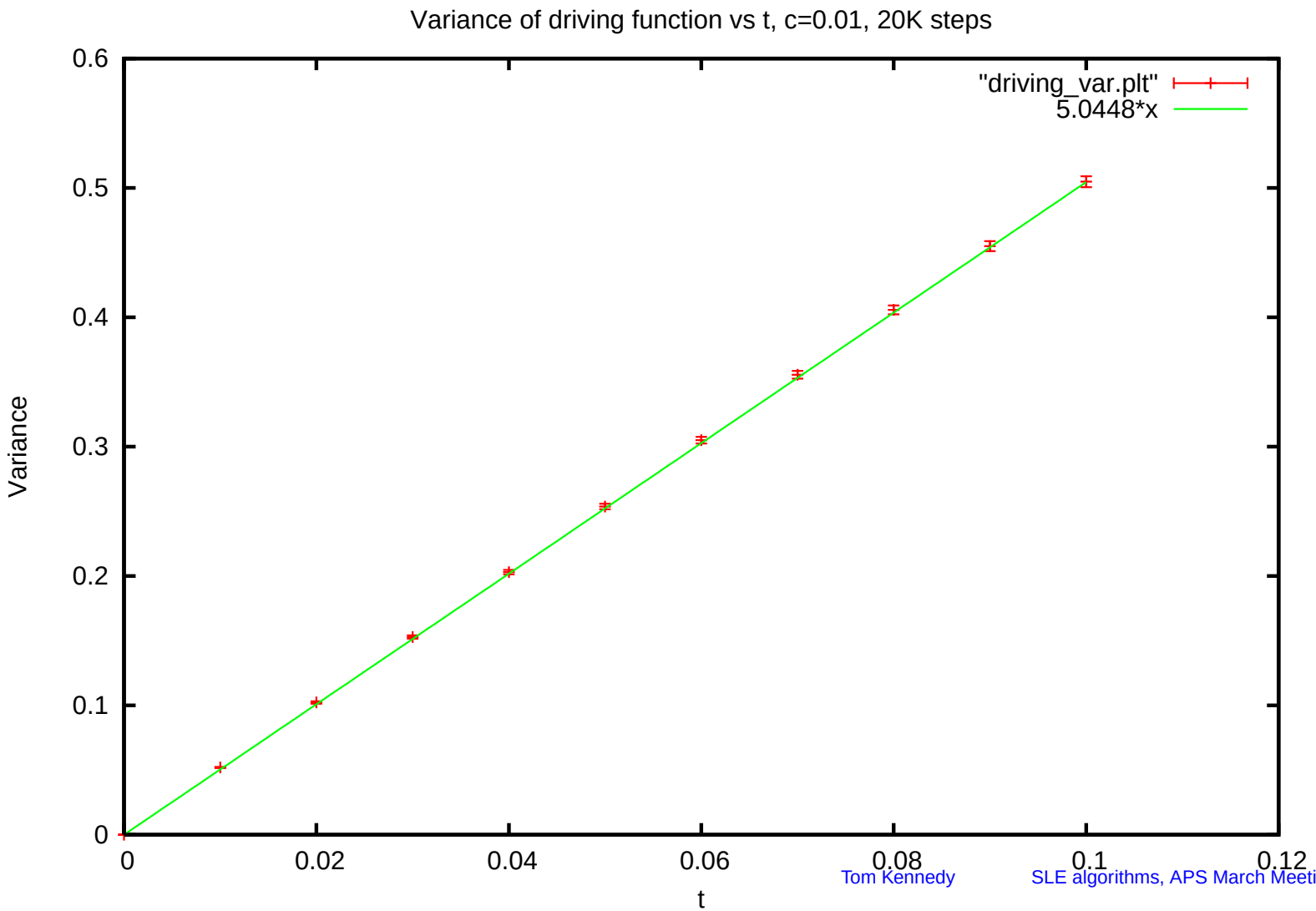
Sample with 50,000 steps, c=0.01



Testing for SLE - gradient-gradient percolation

Variance:

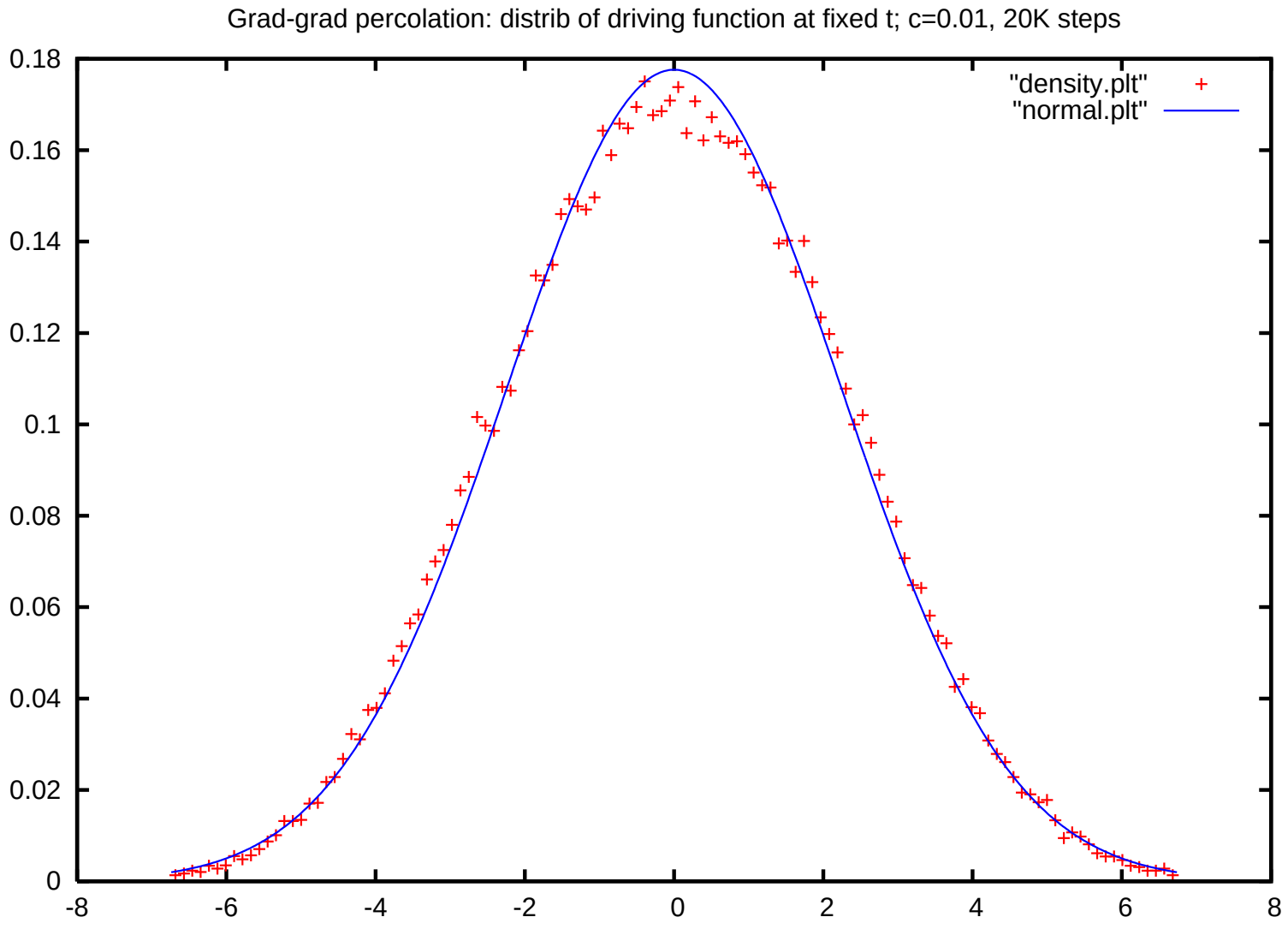
100,000 samples with 20,000 steps, $c = 0.01$
Variance vs. t. Slope (5.0488) estimates "κ."



Testing for SLE - gradient-gradient percolation

Normality:

Distribution of U_t at fixed t vs. normal.



Testing for SLE - gradient-gradient percolation

KS tests for normality

samples	$T/2$	T
5,000	0.302097	0.013082
10,000	0.274908	0.005970
20,000	0.025197	0.025839
50,000	0.000327	0.002934
100,000	0.000008	0.000002

χ^2 tests for independence of increments:

samples	χ^2_a	χ^2_b	χ^2_c
5,000	0.697344	0.374541	0.000000
10,000	0.100552	0.151458	0.000000
20,000	0.247112	0.039399	0.000000
50,000	0.975360	0.000761	0.000000
100,000	0.036366	0.000000	0.000000

Conclusions/Caveats

Unzipping - practical issues:

Difference between using vertical slits vs. tilted slits for the elementary conformal maps h_j is small.

Given that the vertical slit map is considerably faster and easier to implement, we see no reason to use the tilted slit map.

The speed of this algorithm can be increased dramatically using power series approximations of certain analytic functions.

The loss of accuracy from this series approximation is extremely small, insignificant compared to the effect of changing the number of points used to define the curve we are unzipping or compared to the difference between using vertical slits or tilted slits.

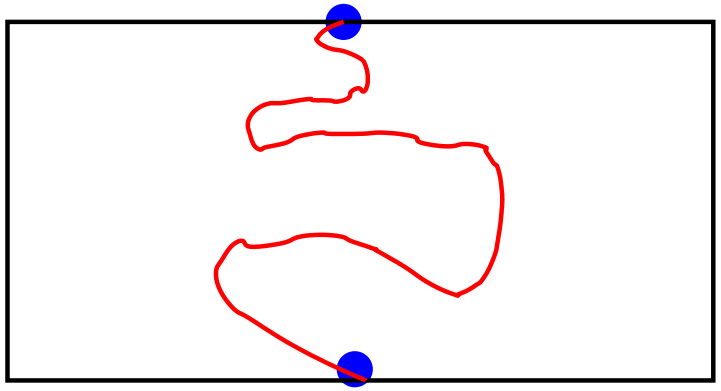
C++ code implementing faster algorithm is available at http://www.math.arizona.edu/~tgk

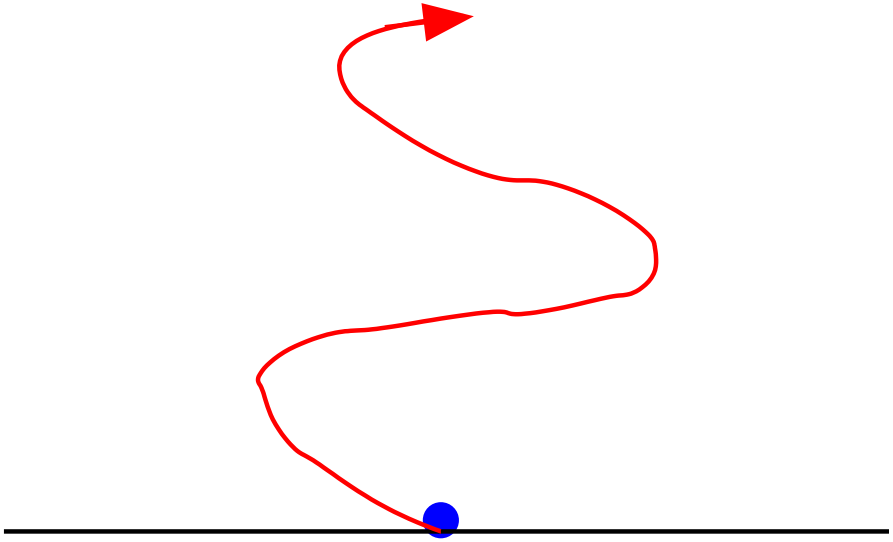
Conclusions/Caveats

Caveat - geometry:

Everything was for curves in half plane from 0 to ∞.

For other simply connected domains, need to conformally map to this:





Conclusions/Caveats

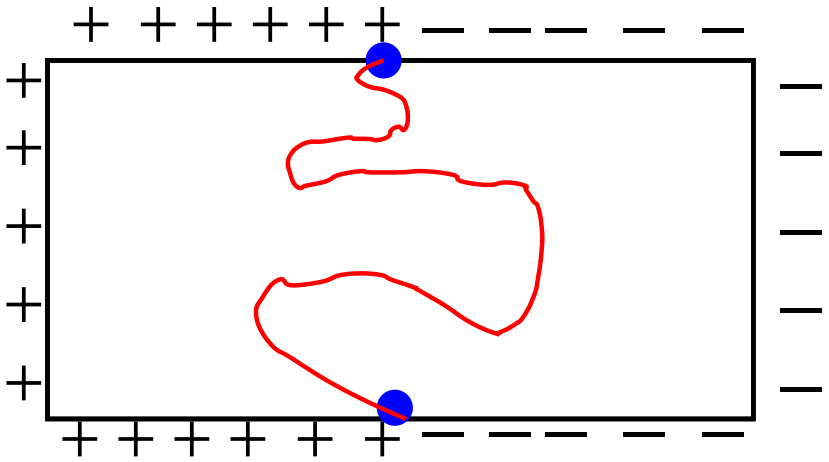
Caveat - boundary conditions:

Need certain boundary conditions to get chordal SLE.

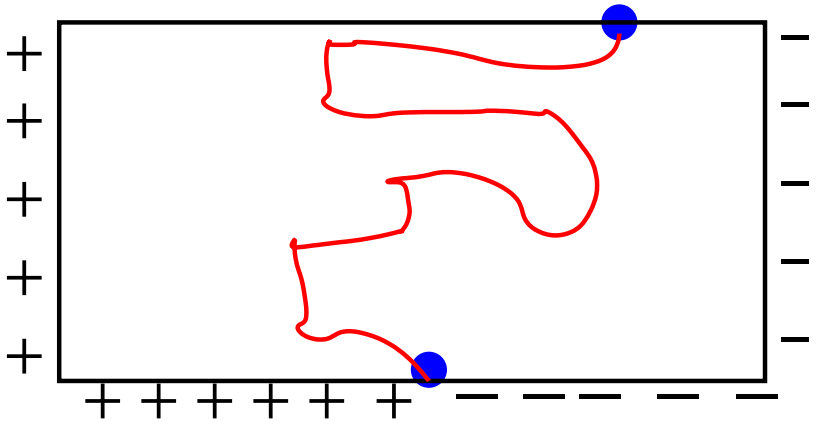
Curve must run between two fixed points on boundary

Ising model: Dobrushin boundary conditions give chordal SLE.

Other boundary conditions give dipolar SLE:



Chordal



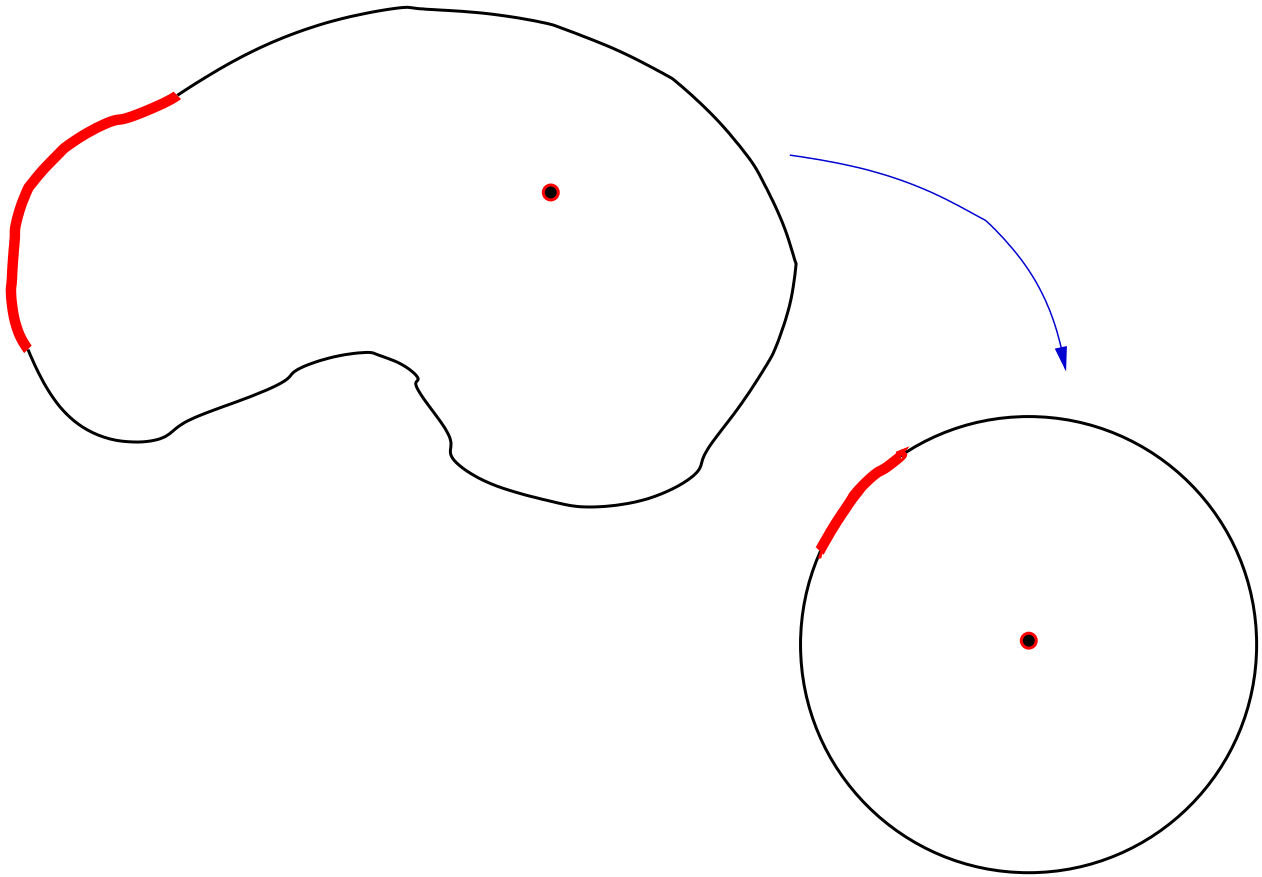
Dipolar

Conclusions/Caveats

Caveat - conformal invariance/covariance

Some things may not be conformally invariant

SAW starting at fixed interior point, ending anywhere on the boundary



This is not SLE. It is a sort of integrated SLE.

Conclusions/Caveats

Conclusions:

- Variance linear in t and U_t normal is not enough!
- "Best" statistical test for Brownian motion depends on model.
- Hard to distinguish something close to SLE from SLE.
  SAW with 5% distortion: need 100,000 samples.

Future work - look at other tests of SLE :

- Explicit computations for SLE are rare.
- Fractal dimension ($= 1 + \kappa/8$),
- Probability of passing right: for a fixed point in half plane, the probability the curve goes right of it is known.
- Compare with simulation of SLE. Time to compute an SLE curve with N points comparable to time to unzip a curve with N points.