Numerical simulation of random curves - lecture 1

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Introduction

Theme: simulations that are related to Schramm-Loewner Evolution (SLE) in some way.

Pedagogic- Ask questions! tgk@math.arizona.edu

Goal: Enable the participants to do state of the art simulations related to SLE.

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- Inverse problem: random curves $\Rightarrow$ driving function
- Lattice models related to SLE: LERW, percolation, SAW, Ising

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On-line at http://www.math.arizona.edu/~tgk

- These slides
- More detailed notes (pdf)
- Computer code (C++/linux)

2 Simulations involving conformal maps

Conformal map: from domain D to domain D'

Bijection that preserves angles

Bijection that is analytic function

Riemann mapping theorem If D, D' are simply connected domains, there is a conformal map between them.

Simply connected means no holes in domain

Not unique, 3 real degrees of freedom

Boundary of domain need not be smooth

2.1 Loewner equation - crash course

$$\mathbb{H} = \{z : Im(z) > 0\}$$

Let γ be a curve in $\mathbb{H}$, starting at 0, with no self intersections (simple).

$\mathbb{H} \setminus \gamma[0, t]$ is a simply connected domain.
$\Rightarrow$ conformal map g_t from it onto $\mathbb{H}$

Simple curves $\Leftrightarrow$ one parameter families of conformal maps.

Parameterize so that

$$g_t(z) = z + \frac{2t}{z} + O(\frac{1}{|z|^2}), \qquad z \to \infty$$

Then Loewner's equation says

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z$$

for some real valued function U_t on $[0, \infty)$, the driving function.

Loewner equation - continued

Tip of curve $\leftrightarrow$ driving function

$$g_t(\gamma(t)) = U_t, \quad \gamma(t) = g_t^{-1}(U_t)$$

Need a limit here.

$\gamma(0) = 0 \Rightarrow U_0 = 0$.

More generally, $\gamma(0) = U_0$.

Smoothness of U_t: Solution to Loewner need not exist for all times t.

Let K_t be the set of points z in $\mathbb{H}$ for which the solution to this equation no longer exists at time t.

Nice $U_t \Rightarrow K_t$ is a curve $\gamma[0, t]$.

But even for a continuous U_t it need not be.

Stochastic Loewner equation

Random curves $\Rightarrow$ random U_t, i.e., stochastic process.

Schramm's discovery was that conformal invariance and a certain "Markov property" imply that U_t must be a Brownian motion with mean zero.

SLE is what you get by taking U_t to be a Brownian motion with mean zero.

For U_t equal to Brownian motion, whether K_t is a curve depends on the variance.

In our simulations our approximation to U_t will be nice enough that it produces a curve.

Loewner equation as sequence of compositions

$g_{t+s} : \mathbb{H} \setminus \gamma[0, t+s] \to \mathbb{H}$.

Do this in two stages:

$g_s : \mathbb{H} \setminus \gamma[0, s] \to \mathbb{H}, \qquad \mathbb{H} \setminus \gamma[0, s+t] \to \mathbb{H} \setminus g_s(\gamma[s, s+t])$,

$\bar{g}_t : \mathbb{H} \setminus g_s(\gamma[s, t+s]) \to \mathbb{H}$ (usual normalization)

Uniqueness $\Rightarrow g_{s+t} = \bar{g}_t \circ g_s, \quad i.e., \quad \bar{g}_t = g_{s+t} \circ g_s^{-1}$, so

$$\frac{d}{dt}\bar{g}_t(z) = \frac{d}{dt} g_{s+t}(g_s^{-1}(z)) = \frac{2}{g_{s+t}(g_s^{-1}(z)) - U_{s+t}} = \frac{2}{\bar{g}_t(z) - U_{s+t}}$$

Note that $\bar{g}_0(z) = z$. Thus $\bar{g}_t(z)$ is obtained by solving the Loewner equation with driving function $\bar{U}_t = U_{s+t}$.

This driving function starts at U_s, and so the curve $\bar{\gamma}(t)$ associated with $\bar{g}_t$ starts growing at U_s.

Loewner eq. as sequence of compositions - cont.

Partition the time interval $[0, \infty)$: $0 = t_0 < t_1 < t_2 < \cdots t_n < \cdots$

$$\bar{g}_k = g_{t_k} \circ g_{t_{k-1}}^{-1}$$

$$g_{t_k} = \bar{g}_k \circ \bar{g}_{k-1} \circ \bar{g}_{k-2} \circ \cdots \cdots \bar{g}_2 \circ \bar{g}_1$$

$\bar{g}_k(z)$ is obtained by solving the Loewner equation with driving function $U_{t_{k-1}+t}$ for $t = 0$ to $t = \Delta_k$, where $\Delta_k = t_k - t_{k-1}$.

$\bar{g}_k$ maps $\mathbb{H}$ minus a curve starting at $U_{t_{k-1}}$ to $\mathbb{H}$.

Intervals of driving function $\leftrightarrow$ composition of maps

It is convenient to shift by $U_{t_{k-1}}$ so our curves start at 0.

Loewner eq. as sequence of compositions - cont.

Define

$$g_k(z) = \bar{g}_k(z + U_{t_{k-1}}) - U_{t_{k-1}},$$

It is obtained by solving the Loewner equation with driving function $U_{t_{k-1}+t} - U_{t_{k-1}}$ for $t = 0$ to $t = \Delta_k$.

Driving function goes from 0 to $\delta_k = U_{t_k} - U_{t_{k-1}}$.

g_k takes $\mathbb{H}$ minus a curve starting at the origin onto $\mathbb{H}$.

and maps tip of that curve to δ_k.

The key idea

Two types of simulations:

- Given driving function, find the curve it generates.
- Given a curve, find the corresponding driving function.

For both problems the key idea is the same.

Approximate the driving function on the interval $[t_{k-1}, t_k]$ by a function for which the Loewner may be explicitly solved.

Maps $\bar{g}_k$ and g_k can then be found explicitly.

g_t is approximated by the composition of the appropriate maps.

Need some explicit solutions to Loewner equation

Tilted slits

$$f(z) = (z + x_l)^{1-\alpha}(z - x_r)^{\alpha},$$

maps $\mathbb{H}$ to $\mathbb{H} \setminus \Gamma$ where Γ is a line segment from 0 to a point $re^{i\alpha\pi}$.
It maps $[-x_l, x_r]$ onto Γ.

Unfortunately, its inverse cannot be explicitly computed.

For normalization we need $(1-\alpha)x_l = \alpha x_r$.

$$f_t(z) = \left(z + 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \left(z - 2\sqrt{t}\sqrt{\frac{1-\alpha}{\alpha}}\right)^{\alpha}$$

produces slit with capacity $2t$.

Tilted slits - continued

$g_t = f_t^{-1}$ solves Loewner eq. with driving function

$$U_t = c_\alpha \sqrt{t}$$

where

$$c_\alpha = \frac{2(1-2\alpha)}{\sqrt{\alpha(1-\alpha)}}$$

NB: $\alpha\pi$ is usual polar angle. Lawler's α is my $1-\alpha$.

Vertical slits

Let

$$g_t(z) = \sqrt{(z-\delta)^2 + 4t} + \delta$$

Then it is easy to check g_t satisfies Lowener's equation with a constant diving function: $U_t = \delta$.

Since the driving function does not start at 0, the curve will not start at the origin.

The curve is just a vertical slit from δ to $\delta + 2\sqrt{t}i$.

Vertical slits mean that we approximate the driving function by a discontinuous piecewise constant function.

Result is not a curve.

2.2 Simulating SLE

Recall that $g_t(\gamma(t)) = U_t$. Define $z_k = g_{t_k}^{-1}(U_{t_k})$. So

$$z_k = \bar{g}_k^{-1} \circ \bar{g}_{k-1}^{-1} \circ \cdots \bar{g}_1^{-1}(U_{t_k})$$

Recall solving Loewner with driving function $U_{t_{k-1}+t} - U_{t_{k-1}}$ for $t = 0$ to $t = \Delta_k$, gives $g_k(z)$ where $g_k(z) = \bar{g}_k(z + U_{t_{k-1}}) - U_{t_{k-1}}$. Define

$$h_k(z) = g_k(z) - \delta_k = \bar{g}_k(z + U_{t_{k-1}}) - U_{t_k}$$

where $\delta_k = U_{t_k} - U_{t_{k-1}}$. Then

$$h_k \circ h_{k-1} \circ \cdots \circ h_1(z_k) = \bar{g}_k \circ \bar{g}_{k-1} \circ \cdots \circ \bar{g}_1(z_k) - U_{t_k} = 0$$

Define $f_k = h_k^{-1}$ so

$$z_k = f_1 \circ f_2 \circ \cdots \circ f_k(0)$$

Simulating SLE - continued

g_k maps $\mathbb{H}$ minus a curve that starts at 0 onto $\mathbb{H}$, sending the tip of the curve to δ_k.

So h_k maps $\mathbb{H}$ minus the curve onto $\mathbb{H}$, sending the tip to 0.

So $f_k = h_k^{-1}$ maps $\mathbb{H}$ onto $\mathbb{H}$ minus the curve, sending 0 to the tip.

The first map f_k welds together a small interval on $\mathbb{R}$ containing the origin to produce a small cut.

Origin is mapped to the tip of this cut.

The second map f_{k-1} welds together a (possibly different) small interval to produces another small cut.

The original cut is moved away from the origin with its base being at the tip of the new cut.

This process continues. Each map introduces a new small cut whose tip is attached to the image of the base of the previous cut.

Choice of δ_k, Δ_k

$$\Delta_k = t_k - t_{k-1}, \qquad \delta_k = U_{t_k} - U_{t_{k-1}}$$

We approximate U_t on each time interval $[t_{k-1}, t_k]$ so that $g_k(z)$ is known explicitly.

The two constraints on g_k are that the curve must have capacity $2\Delta_k$ and g_k must map the tip of the curve to δ_k.

Different choices of how we choose g_k subject to these constraints give us different discretizations of the curve.

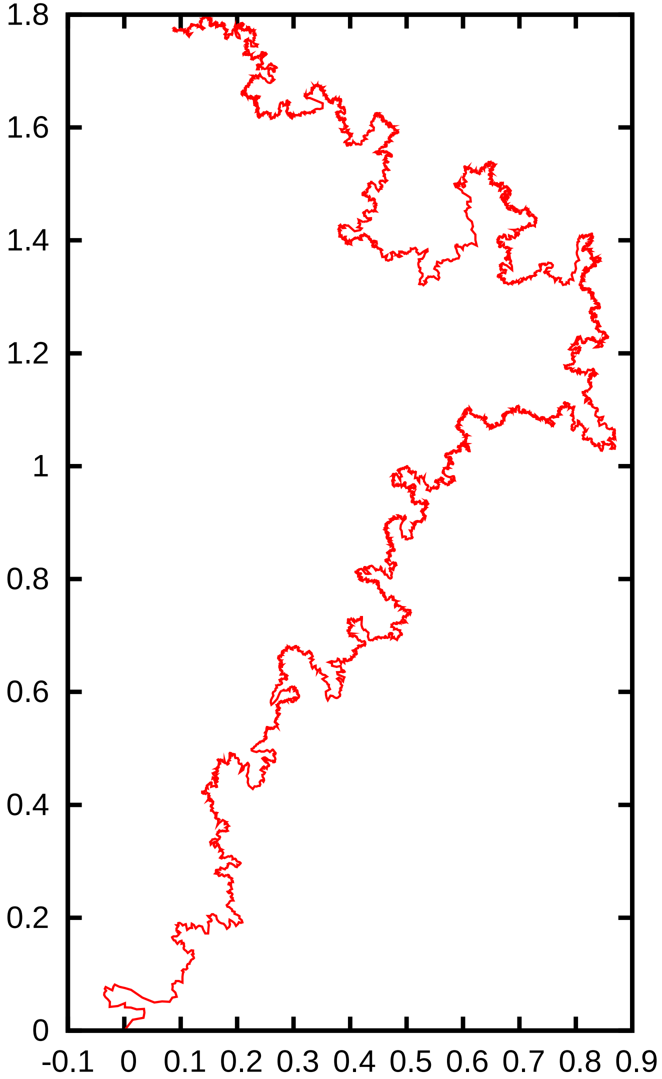
However, this choice will not have a significant effect.

Of much greater importance is how we choose the Δ_k and δ_k.

Uniform Δ_k

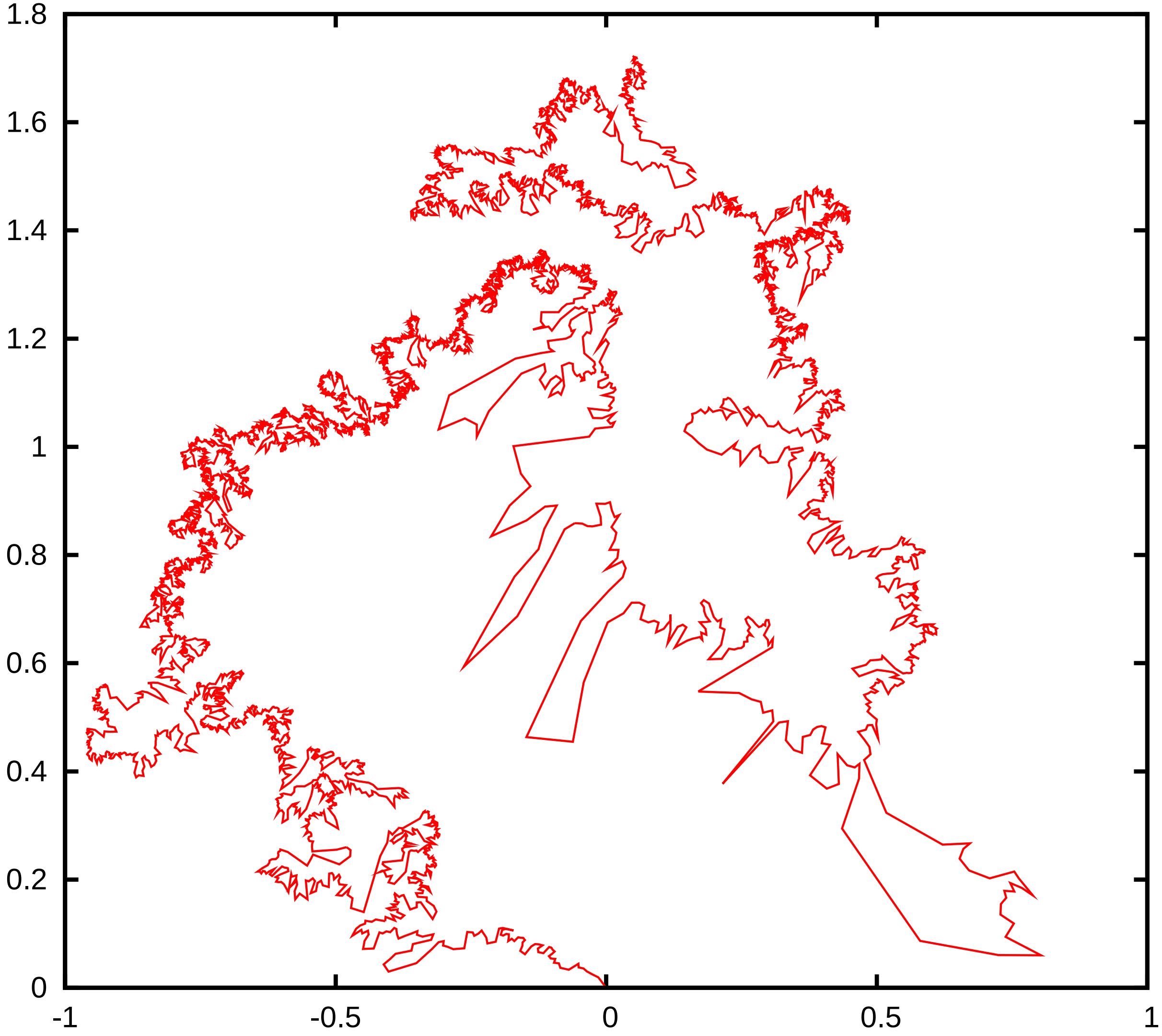
Uniform Δ_k is simplest. For κ not too large this works reasonably well

Figure shows a curve with $\kappa = 8/3$ and $N = 10,000$ points.



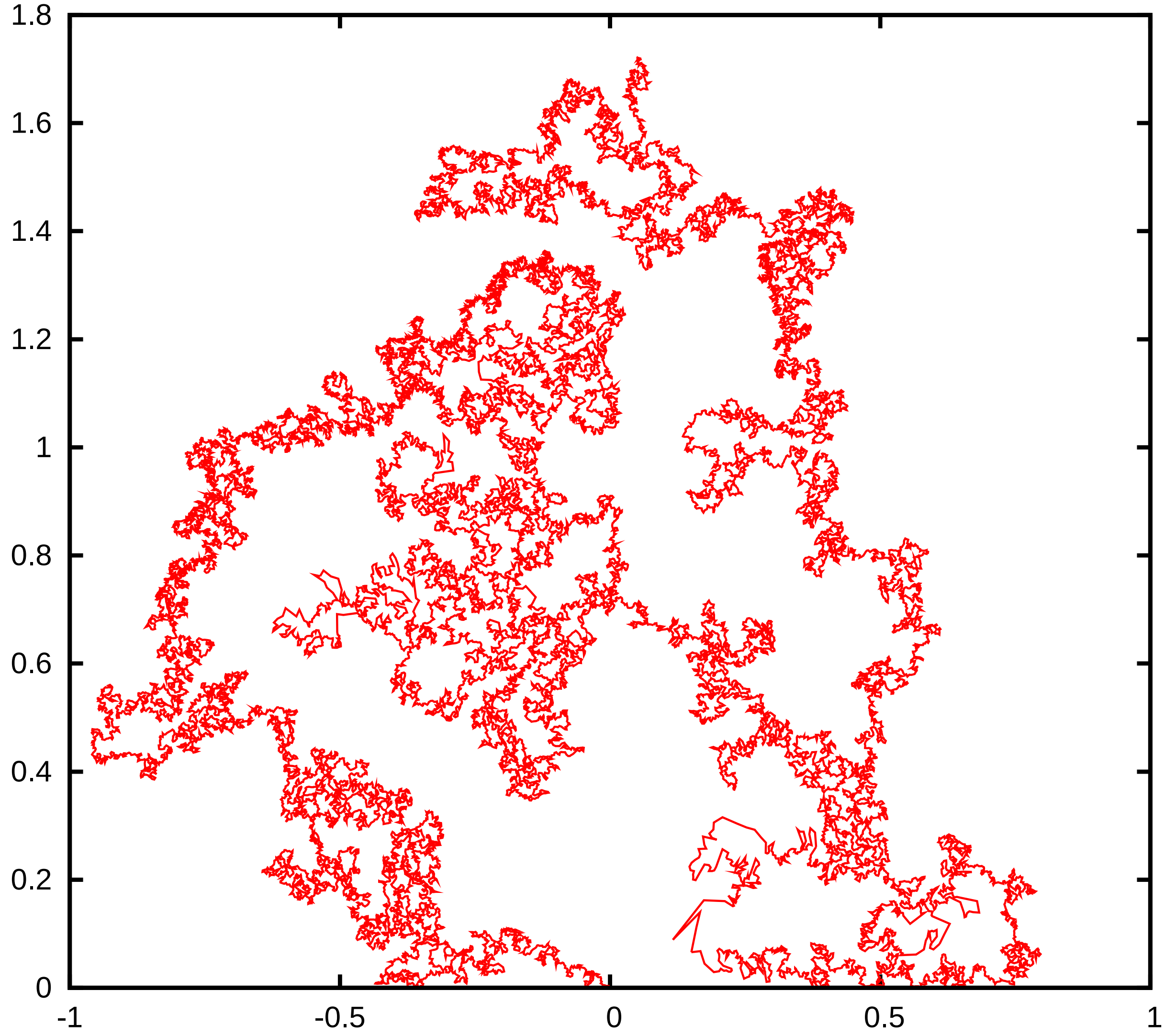
Uniform Δ_k - continued

For larger values of κ, uniform Δ_k is a disaster. Figure shows $\kappa = 6$.



Adaptive Δ_k

For larger values of κ, we should use a varying Δ_k. (Rhode).



Capacity is not "length"

The (half plane) capacity C of a set A is

$$g(z) = z + \frac{C}{z} + O(\frac{1}{z^2})$$

where g maps $\mathbb{H} \setminus A$ onto $\mathbb{H}$, usual normalizations.

More intuitive def:

$$C = \lim_{y \to \infty} y \, E^{iy}[Im(B_\tau)]$$

where B_t is 2d Brownian motion started at iy

τ is the first time it hits A or $\mathbb{R}$.

Adaptive Δ_k

Fix a spatial scale ϵ.

Start with uniform Δ_k. Compute z_k.

Look for z_k such that $|z_k - z_{k-1}| \geq \epsilon$

For these time intervals $[t_{k-1}, t_k]$, divide the interval into two equal intervals.

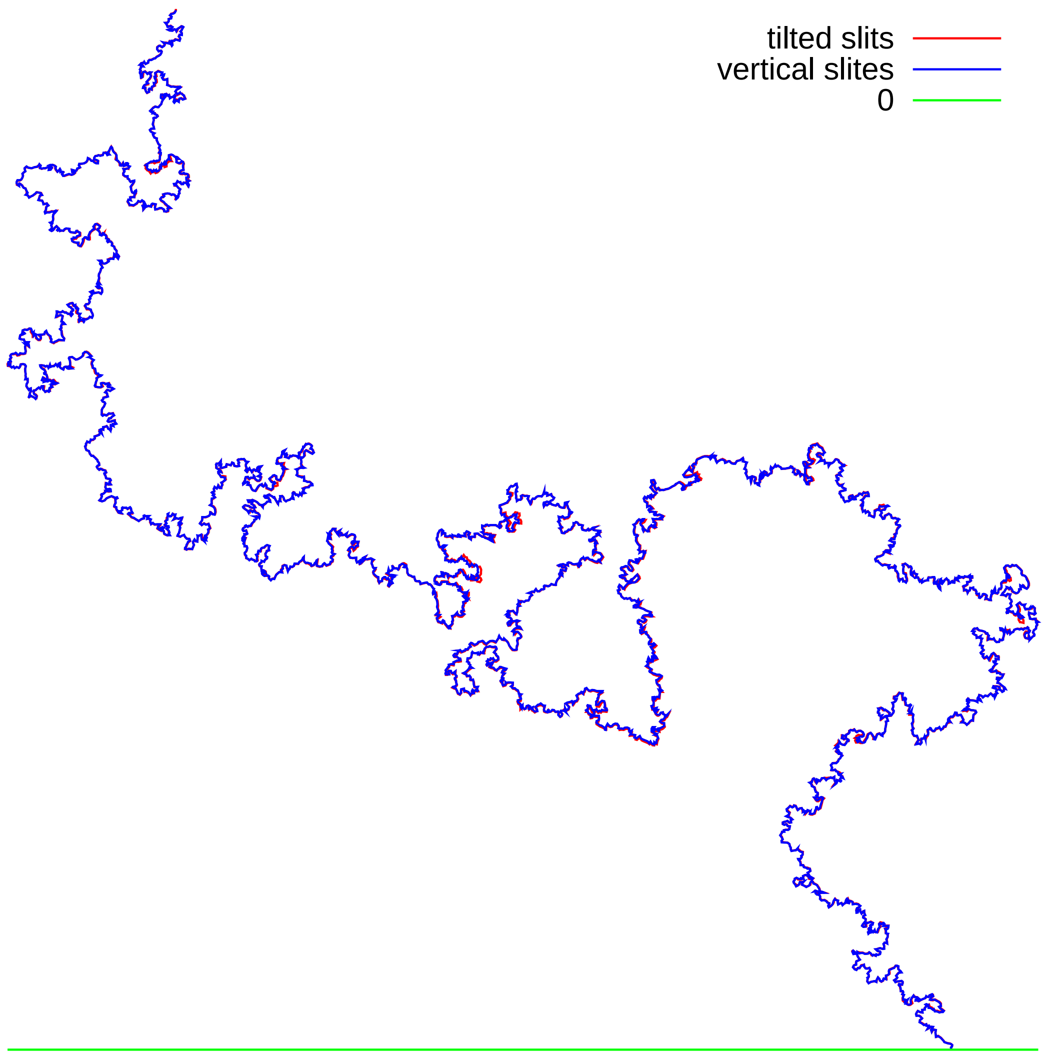
Sample Brownian motion at midpoint of $[t_{k-1}, t_k]$ using Brownian bridge.

Compute new z_k, $\cdots$.

Repeat until all $|z_k - z_{k-1}| \leq \epsilon$.

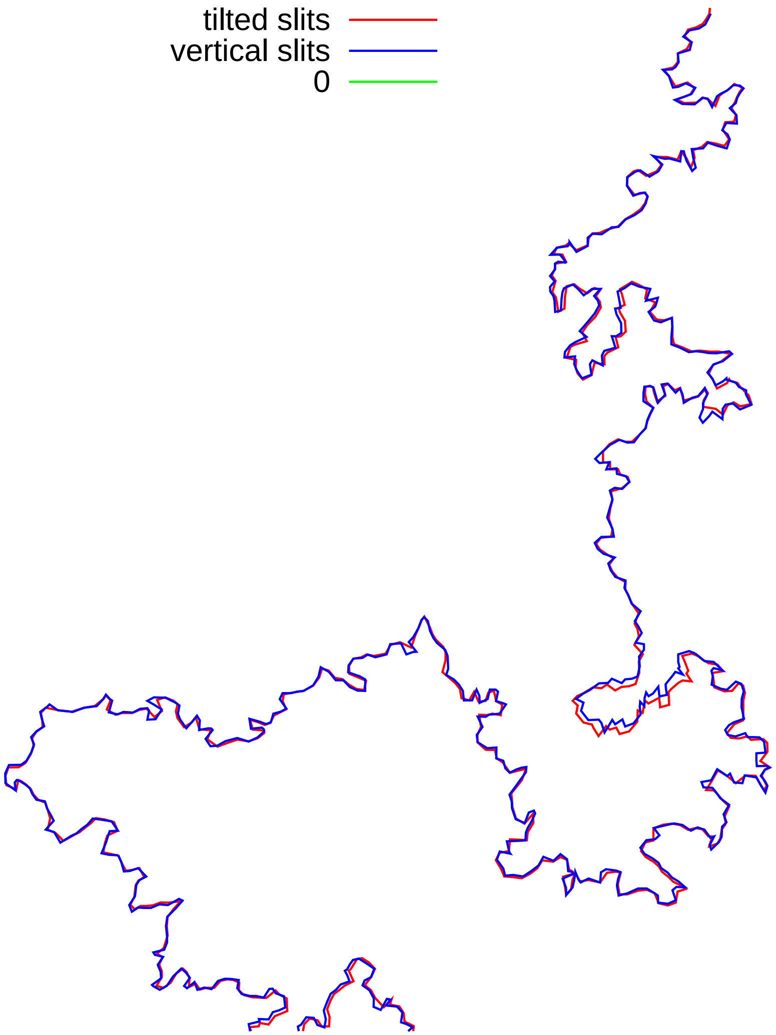
Effect of choice of cut

$\kappa = 8/3$, tilted slits vs. vertical slits, about $35,000$ points



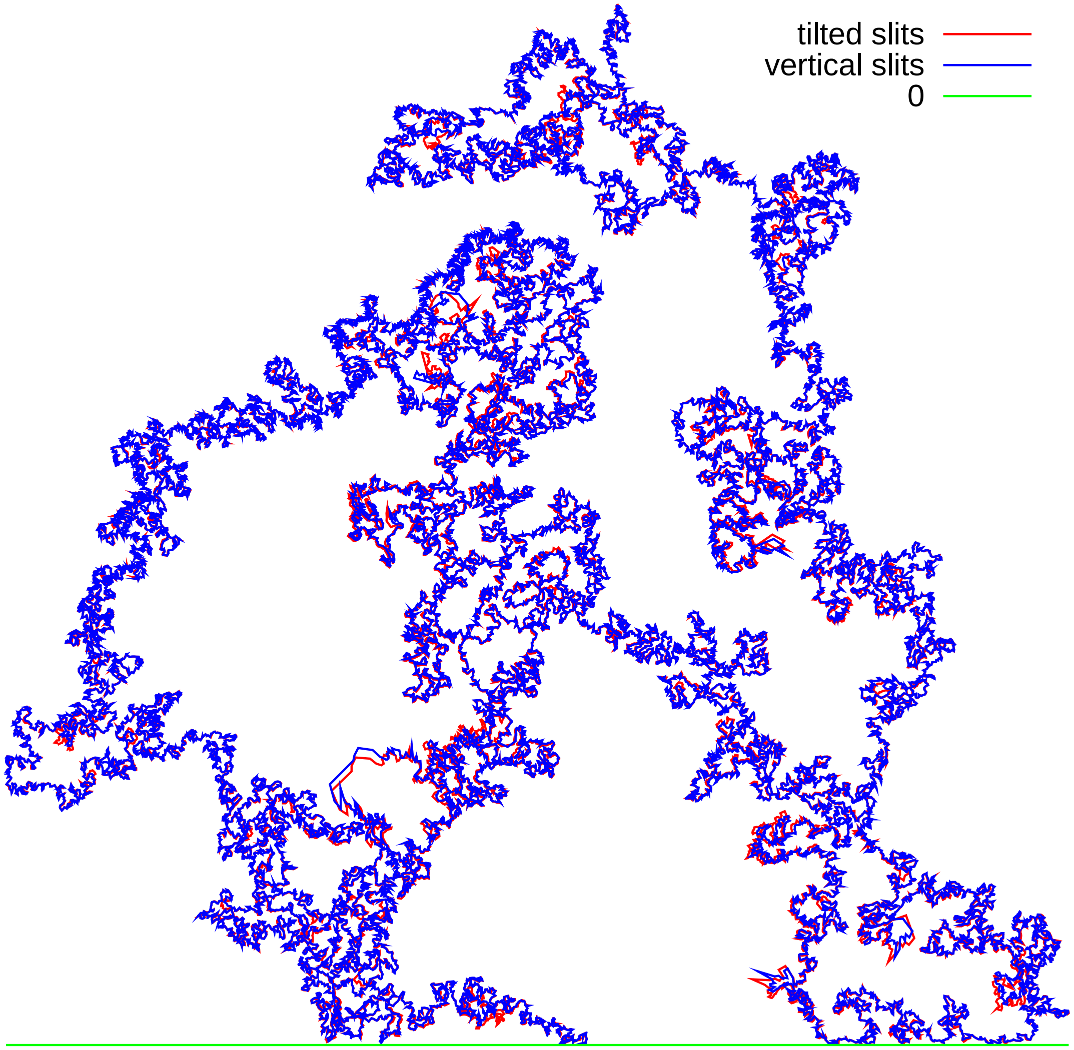
Effect of choice of cut - continued

$\kappa = 8/3$, tilted slits vs. vertical slits - enlarged.



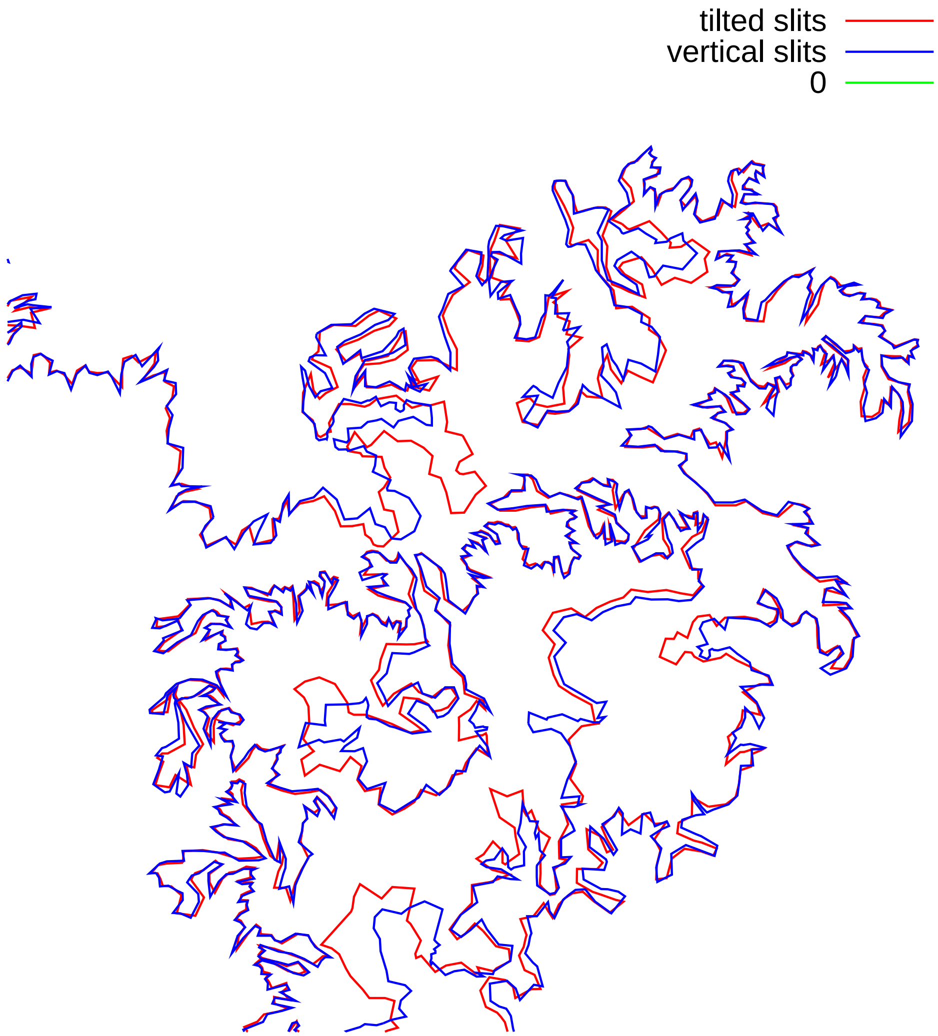
Effect of choice of cut

$\kappa = 6$, tilted slits vs. vertical slits, about $35,000$ points



Effect of choice of cut - continued

$\kappa = 6$, tilted slits vs. vertical slits - enlarged.



2.3 Inverse SLE: curve $\Rightarrow$ driving function

Given curve, compute the driving function.

Why would you want to do this?

Usually the parameterization of the curve is not by capacity.

So if $g_s : \mathbb{H} \setminus \gamma[0,s] \to \mathbb{H}$ then

$$g_s(z) = z + \frac{C(s)}{z} + O(\frac{1}{z^2}),$$

Coefficient $C(s)$ is capacity of $\gamma[0,s]$.

The value of the driving function at t is $U_t = g_s(\gamma(s))$, $2t = C(s)$.

Algorithm for finding the conformal map is “zipper” algorith.

Shifting conformal maps - philosophy

We find it more convenient to work with the conformal map

$$h_s(z) = g_s(z) - U_t$$

where $C(s) = t$.

$h_s : \mathbb{H} \setminus \gamma[0, s] \to \mathbb{H}$, sends tip of $\gamma(s)$ to the origin.

The value of the driving function at s is minus the constant term in the Laurent expansion of h_s about ∞:

$$h_s(z) = z - U_t + \frac{2C(s)}{z} + O(\frac{1}{z^2}),$$

Sequence of compositions

Let γ be a curve in $\mathbb{H}$ starting at 0.

Let $z_0, z_1, \cdots, z_n$ be points along the curve with $z_0 = 0$.
In many applications these are lattice sites.

Let $2t_k$ be capacity of curve up to z_k.

Define $\bar{g}_k, g_k, h_k, f_k$ as before, so

$$z_k = f_1 \circ f_2 \circ \cdots \cdots \circ f_{k-1} \circ f_k(0)$$

To simulate SLE we knew the f_k and computed the z_k.

$$h_k \circ h_{k-1} \circ \cdots h_2 \circ h_1(z_k) = 0$$

Now we know the z_k and want to find the h_k.

Finding the h_k

Suppose $h_1, h_2, \cdots, h_k$ have been defined.

So $h_k \circ h_{k-1} \circ \cdots h_2 \circ h_1$ sends z_k to 0

It should send z_{k+1} to a point close to origin :

$$w_{k+1} = h_k \circ h_{k-1} \circ \cdots \circ h_1(z_{k+1})$$

Let γ_{k+1} be a short simple curve that ends at w_{k+1}, e.g., tilted slit or vertical slit.

Let h_{k+1} be corresponding conformal map.

h_{k+1} has two real degrees of freedom, determined by w_{k+1}.

Finding the driving function

Let $2\Delta_i$ be capacity of h_i

Let δ_i be final value of the driving function for h_i. So

$$h_i(z) = z - \delta_i + \frac{2\Delta_i}{z} + O(\frac{1}{z^2})$$

Then

$$h_k \circ h_{k-1} \circ \cdots \circ h_1(z) = z - U_{t_k} + \frac{2t_k}{z} + O(\frac{1}{z^2})$$

where

$$t_k = \sum_{i=1}^{k} \Delta_i, \qquad U_{t_k} = \sum_{i=1}^{k} \delta_i$$

Driving function of the curve is sum of driving functions of the elementary conformal maps h_i.

Zipper algorithm for conformal maps

Digression: original zipper algorithm for computing conformal maps. See Marshall/Rhode paper.

D is simply connected domain. $z_0, z_1, \cdots z_n$ are points on its boundary (counter clock-wise).

$$\phi(z) = i\sqrt{\frac{z - z_1}{z - z_0}}$$

$z_0 \to \infty$, $z_1 \to 0$, $\quad z_2, z_3, \cdots, z_n$ are mapped to curve γ in $\mathbb{H}$.

Plane is mapped to $\mathbb{H}$, $\quad D$ to one side of γ in $\mathbb{H}$.

Unzip γ. $\quad$ At end D is mapped to quarter plane.

Map quarter plane to $\mathbb{H}$ or unit disc or ...

Lecture 2

- Faster algorithm for simulating SLE
- Faster algorithm for computing driving function.
- Results - pictures of SLE
- Results - driving process of self-avoiding walk
- Open problems

Intermission

Show movie at intermission