Numerical simulation of random curves - lecture 2

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Supported by NSF grant DMS-0501168

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2.4 Faster algorithms

Goal: speed up algorithms for

- Computing curve γ given driving function U_t
- Computing driving function U_t given γ

Recall: points on the curve are

$$z_k = f_1 \circ f_2 \circ \cdots \circ f_k(0)$$

Computing single z_k takes k operations

Computing all the points takes $O(N^2)$ operations

Finding the driving function

Note : computation of z_k does not depend on z_j.

You can compute a subset of the points.

For example, just computing the tip takes $O(N)$

We will use the time it takes to compute one point on the curve as the natural measure of how fast an algorithm is.

TPPC= “time per point computed”

TPPC is $O(N)$ for naive algorithm

Our goal : algorithm so that TPPC is $O(N^p)$ with $p < 1$.

Blocking the compositions

Key idea: Group functions in the composition into blocks.

b is number of functions in a block.

Define

$$F_j = f_{(j-1)b+1} \circ f_{(j-1)b+2} \circ \cdots \circ f_{jb}$$

Let $k = mb + l$ with $0 \leq l < b$. Then

$$z_k = F_1 \circ F_2 \circ \cdots \circ F_m \circ f_{mb+1} \circ f_{mb+2} \circ \cdots \circ f_{mb+l}(0)$$

Number of compositions in above is smaller than original composition by factor of b.

Compute each F_j only once.

Computing the blocks

The f_i are simple, but the F_j cannot be explicitly computed.

Approximate the f_i by series whose compositions can be explicitly computed to give an explicit series approximation to F_j.

For large z, $f_i(z) \approx$ its Laurent series about ∞.

Our approximation is slightly different from Laurent series.

Why this will work:

$$f(z) = z + \delta - \frac{2\Delta}{z} + O(\frac{1}{|z|^2})$$

The series expansion

Let $\gamma : [0, t] \to \mathbb{H}$ be curve ($\gamma(0) = 0$). $f : \mathbb{H} \to \mathbb{H} \setminus \gamma[0, t]$.

Let $a, b > 0$ be such that $[-a, b]$ is mapped onto the slit $\gamma[0, t]$.

Then f is real valued on $(-\infty, -a]$ and $[b, \infty)$.

Schwartz reflection principle says f has an analytic continuation to $\mathbb{C} \setminus [-a, b]$, denote by f.

Let $R = \max\{a, b\}$. f is analytic on $\{z : |z| > R\}$ and maps ∞ to itself.

Thus $f(1/z)$ is analytic on $\{z : 0 < |z| < 1/R\}$, simple pole at the origin with residue 1.

$$f(1/z) = 1/z + \sum_{k=0}^{\infty} c_k \, z^k$$

This gives the Laurent series of f about ∞.

$$f(z) = z + \sum_{k=0}^{\infty} c_k \, z^{-k}$$

The series expansion - cont

Instead of Laurent we do following

Define $\hat{f}(z) = 1/f(1/z)$.

Since $f(z) \neq 0$ on $\{|z| > R\}$, $\hat{f}(z)$ is analytic in $\{z : |z| < 1/R\}$.
And $\hat{f}(0) = 0$, $\hat{f}'(0) = 1$.

So $\hat{f}$ has a power series of the form

$$\hat{f}(z) = \sum_{j=0}^{\infty} a_j z^j$$

with $a_0 = 0$ and $a_1 = 1$.

Radius of convergence is at least $1/R$.

Refer to this series as the “hat power series” of f.
It is Laurent series of $1/f$.

Series and composition

Advantage - behavior with respect to composition:

$$(f_1 \circ f_2)\hat{}(z) = 1/f_1(1/\hat{f}_2(z)) = \hat{f}_1(\hat{f}_2(z))$$

Thus

$$(f_1 \circ f_2)\hat{} = \hat{f}_1 \circ \hat{f}_2$$

Our approximation : truncate at order n.

$$f(z) = \frac{1}{\hat{f}(1/z)} \approx \left[\sum_{j=0}^{n} a_j z^{-j} \right]^{-1}$$

For each f_i, compute the hat series to order n.

Use them to compute the hat series of F_j to order n.

Series radius of convergence

Let $1/R_j$ be the radius of convergence for the power series for $\hat{F}_j$.

If z is large compared to R_j, then $F_j(z)$ is well approximated using its hat power series.

Introduce a parameter $L > 1$.

Use the hat power series for $F_j(z)$ when $|z| \geq LR_j$.
Otherwise, just use composition of f_i expression.

Which one depends on z and so is random.

Note : R_j is easy to compute.

R_j is the smallest positive number such that $F_j(R_j)$ and $F_j(-R_j)$ are both real.

Choices for n, L and b

b is number of functions composed in a block

n is order at which we truncate our series approximation

L is scale that determines when we use series for F_j

Choose b for speed, little effect on accuracy.

Best b depends on N, roughly as $c\sqrt{N}$

For SLE simulation, ususally take $b = 20$.

For unzipping SAW with $N = 1,000$ to $500,000$, we use $b = 20$ to 200.

Choice of n involves trade-off of speed vs accuracy. We use $n = 12$.

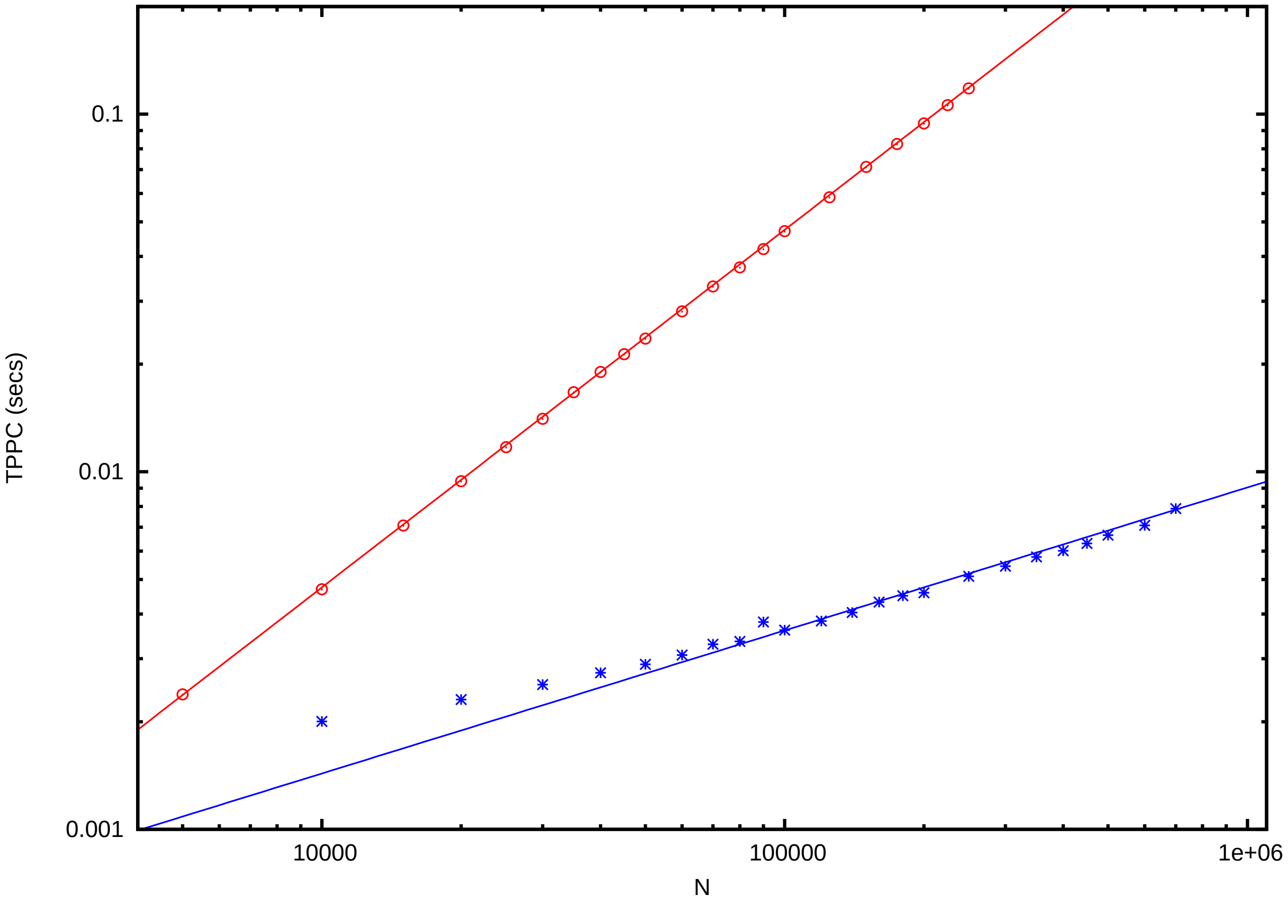
Choice of L involves trade-off of speed vs accuracy. We use $L = 4$.

How fast is it?

TPPC vs N = number of time intervals. $\kappa = 6$.

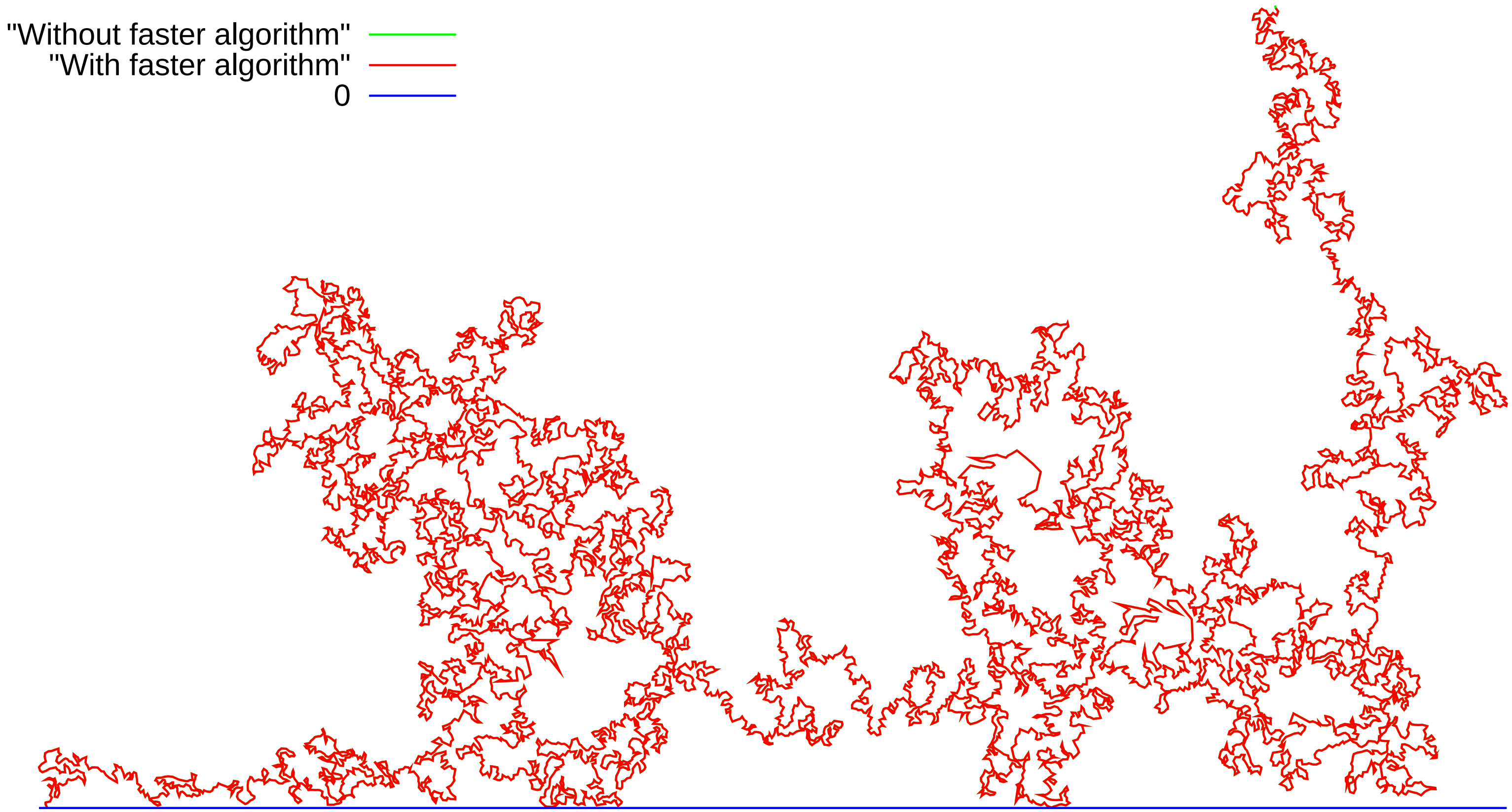
Top curve is naive algorithm; bottom curve is new algorithm.

The lines shown have slopes 1 and 0.4.



How accurate is it?

SLE with $\kappa = 6$ with and without faster algorithm.



Accuracy - continued

SLE with $\kappa = 6$ with and without faster algorithm.

Enlarged

Faster zipper for driving

Now consider algorithm for curve $\Rightarrow$ driving function.

Single w_{k+1} takes $O(k)$ time. All the w_{k+1} takes $O(N^2)$.

As before, group functions we are composing into blocks.

Approximate composition within a block using series.

Minor difference: order of conformal maps.

$$w_{k+1} = h_k \circ h_{k-1} \circ \cdots \circ h_1(z_{k+1})$$

$$H_j = h_{jb} \circ h_{jb-1} \circ \cdots \circ h_{(j-1)b+2} \circ h_{(j-1)b+1}$$

With $k = mb + r$ with $0 \leq r < b$,

$$w_{k+1} = h_{mb+r} \circ h_{mb+r-1} \circ \cdots \circ h_{mb+1} \circ H_m \circ H_{m-1} \circ \cdots \circ H_1(z_{k+1})$$

Approximate h_i by its hat power series and compute the compositions H_j just once.

Faster zipper for driving - radius of convergence

The hat power series is defined as before.

Use it to compute H_j as before. Radius of convergence?

Let h be one of the h_i.

It maps $\mathbb{H}$ minus a simple curve near the origin to $\mathbb{H}$.

Let R be the largest distance from the origin to the curve.

Then h is analytic on $\{z \in \mathbb{H} : |z| > R\}$.

h is real valued on the real axis, so Schwarz reflection principle $\Rightarrow$ analytic continuation to $\{z \in \mathbb{C} : |z| > R\}$.

h does not vanish on this domain, let $\hat{h}(z) = 1/h(1/z)$.

$$\hat{h}(z) = \sum_{j=1}^{\infty} a_j z^j$$

The radius of convergence of this power series is $1/R$.

Computing radius of convergence

We need to compute R_j.

Consider images of $z_{(j-1)b+1}, z_{(j-1)b+2}, \cdots z_{jb}$ under the map $H_{j-1} \circ H_{j-2} \circ \cdots \circ H_1$.

$$H_j : \mathbb{H} \setminus \Gamma_j \to \mathbb{H}$$

where Γ_j is some curve which passes through these images.

R_j is the maximal distance from the origin to Γ_j.

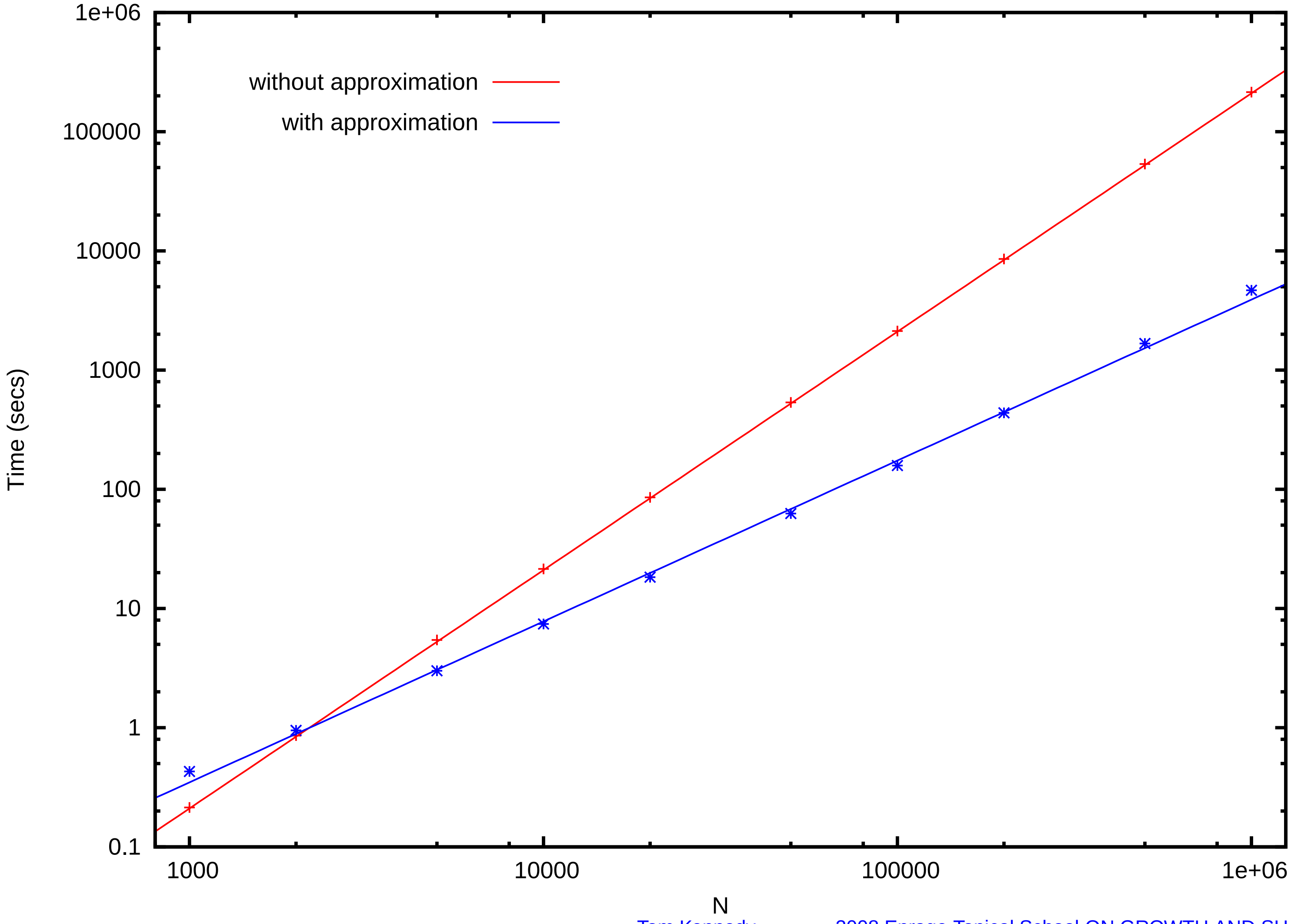
Approximate this by maximum distance from the origin to the images of $z_{(j-1)b+1}, z_{(j-1)b+2}, \cdots z_{jb}$ under $H_{j-1} \circ H_{j-2} \circ \cdots \circ H_1$.

How fast is it?

Unzip a SAW. Plot time (in seconds) vs. N, number of step.

Optimize block size b

Lines have slope 2 and 1.35.



Pictures of SLE

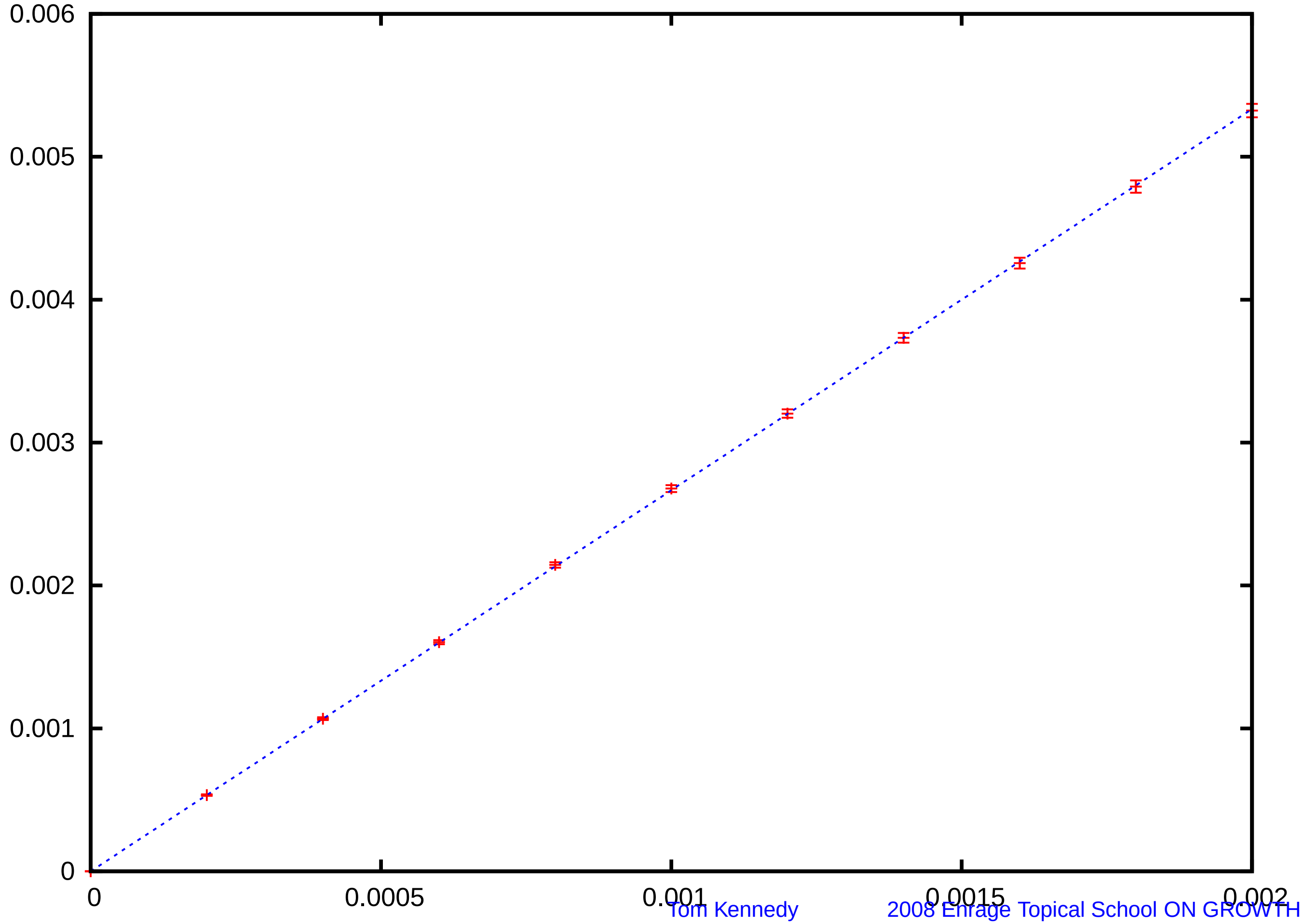
firefox web_sle/index.html

Self-avoiding walk driving process

SAW with 200,000 steps; only compute driving function up to a fixed capacity T. Average number of steps in “unzipped” portion is 9350.

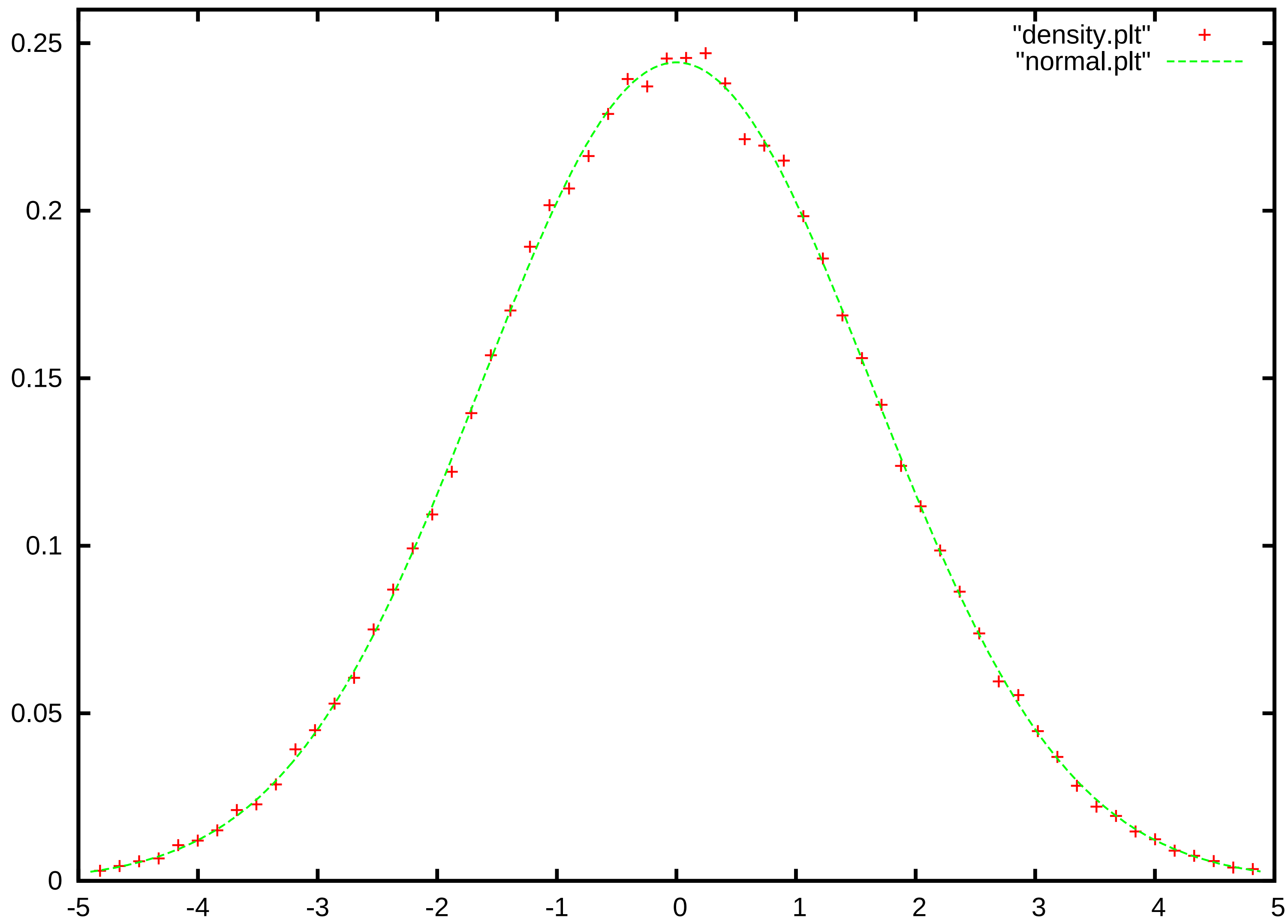
100,000 samples, i.e., SAW’s

Plot variance of driving process, compare slope to $8/3$: 2.6686 ± 0.0132.



Self-avoiding walk driving process -cont

Distribution of driving function at T.



Computing the hat power series

We consider how to compute the hat power series, first for the problem of computing the curve given the driving function.

Tilted slits at an angle $\alpha\pi$.

$$f(z) = (z + x_l)^{1-\alpha}(z - x_r)^{\alpha},$$

$$\hat{f}(z) = z\,(1 + x_l z)^{-(1-\alpha)}\,(1 - x_r z)^{-\alpha}$$

The power series of the last two factors:

$$(1 - cz)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\, c^k\, z^k$$

Computing the hat power series - cont

Vertical slits

$$f(z) = \sqrt{z^2 - 4t} + x,$$

First consider $s(z) = \sqrt{z^2 - 4t}$.

$$\hat{s}(z) = \frac{z}{\sqrt{1 - 4tz^2}} = z \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!} 2^k t^k z^{2k}$$

Note $f = t \circ s$ with $t(z) = z + c$.
So the hat power series of f is just the composition of the hat power series for t and s. The series for t :

$$\hat{t}(z) = \frac{1}{1/z + c} = \frac{z}{1 + cz} = z \sum_{m=0}^{\infty} (-1)^m c^m z^m$$

Computing the hat power series - cont

Now consider the problem of computing the driving function given the curve

Tilted slits $h(z) = f^{-1}(z)$ cannot be found explicitly. So we compute the series for f and then invert the series.

Vertical slits $h(z) = f^{-1}(z)$ has the same structure as f and so computing the series is straightforward.

Multiplication, composition: We also need routines to

- Multiply two power series
- Compose two power series

Straightforward - they are just power series.

Open problems/projects - homework

- Radial SLE: can you use these ideas to make this fast?
- Fingerprint for loops.
- Can you learn anything about the driving process for near-critical models (massive scaling limits)?
- Is the faster algorithm a competitive way to compute conformal maps?
- SLE(κ,ρ) simulation
- More user friendly implementation of code (Matlab?)
- Audience?