The Exponential Function and the Dynamics of Population, Part 1

by William Yslas Velez and Catherine M. Yslas
University of Arizona

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The Southwest Regional Institute in the Mathematical Sciences (SWRIMS) based at the University of Arizona was a National Science Foundation project whose purpose was to shed light on how university researchers could become involved in the education of the nation. One outcome of SWRIMS was a series of four activities related to the dynamics of population growth. The introductory activities in this issue include students' reactions to the investigations. These will be followed by the concluding activities in the next issue.

Activity 1: Powers of Two

We want students to personally experience the exponential function—both growth and decay—in a variety of ways. Students are asked to perform activities that generate powers of 2, both growing (1, 2, 4, 8, . . .) and decaying (1, 1/2, 1/4, 1/8, . . .). In the first activity students simply fold a sheet of paper in half and gather data on the thickness after each fold. For most pre-algebra students this process may involve counting the layers or numbering the rectangles before they recognize the pattern. After a few folds many advanced students write out a table of 10 or more folds. Other students will write, "after n folds, the number of layers is 2^n." The pre-algebra students ask, quite correctly, "What does it mean to have predicted 1024 layers with 10 folds when I can't even fold the paper in half eight successive times?" The ensuing discussion should give the students a first glimpse into the applicability and the drawbacks of a mathematical model.

folds	area	area	area	area	area
0	1	93.50	603.00	100.00	603
1	1/2	46.75	301.50	50.00	297
2	1/4	23.38	150.75	25.00	152
3	1/8	11.69	75.38	12.50	73
4	1/16	5.84	37.69	6.25	30
5	1/32	2.92	18.84	3.63	17

Table 1

A more varied collection of responses comes from students who are asked to measure the area of one of the 2^n regions after n successive folds. Some examples are given in Table 1.

In the first four columns, we see that the student groups are each making a different choice of units: 1 sheet of paper = 93.5 square inches = 603 square centimeters = 100 units. The groups explained their own lists by describing a mental model that says to divide the area by two and enter each result on the table beneath the previously computed value. The last column in the table shows one group's actual measurements in square centimeters on one of the folded regions of paper. This data driven point of view allows for the continuation of the discussion of the value of mathematical models. Rarely are scientists given both the data and a mental model. Frequently, the investigator has only measurements like those tallied in the last column and the task is to find an appropriate mathematical function to model the data.

A second introductory activity to model exponential growth and decay uses a distance probe and a calculator or computer to give a graph of distance between a moving person and a probe. The ultimate goal is to walk the functions 2^t and 2^{-t} where t measures time in seconds. After a few demonstrations, students quickly catch on and are able to describe the motion associated with the properties of a graph traditionally discussed in calculus class.

Activity 2: Random Models of Growth and Decay

This classroom activity is based on two concurrently running experiments—one of the growth of a bacteria culture and a second on coin tossing models. For some students, the connection between mathematics and biology is the best part of the whole exponential experience. Only the coin-tossing experiment will be described here.

The decay model begins with the distribution of 100–200 pennies per group. Enter that number in generation 0. Flip each group of coins once. Discard the heads, count the remainder, and enter that number as generation 1. Continue this process until all of the coins are discarded. Table 2 is an example of a table that students created.

generation	coins	coins	coins	coins	coins	coins	coins	coins
0	184	154	183	132	164	122	198	121
1	97	79	98	64	80	67	84	43
2	44	36	46	29	34	44	41	22
3	20	16	21	11	19	21	20	11
4	14	7	11	4	9	9	13	8
5	8	4	6	2	3	7	6	3
6	3	1	2	2	3	2	4	3
7	1	0	1	0	2	1	3	1
8	1		0		0	1	3	1
9	1					1	3	1
10	1					0	2	0
11	0						1	
12							0	

Table 2

The growth model begins by tossing a single coin. If the coin lands tails, add no more coins and enter the number 1 for the number of coins in generation 1. If the coin lands heads, add two more coins and enter the number 3 for the number of coins in generation 1. For generation 2, take all of the coins in generation 1, flip them and add 2 coins for every head landed. Continue tossing coins until the sixth (or more) generation data is gathered. Another student generated table is included in Table 3.

generation	coins	coins	coins	coins	coins	coins	coins	coins
0	1	1	1	1	1	1	1	1
1	1	3	3	3	3	1	1	1
2	3	7	5	9	7	3	3	1
3	3	11	9	19	11	7	5	3
4	9	21	15	35	17	19	9	5
5	19	35	31	71	31	37	15	9
6	41	67	63	157	69	83	31	23
7	85	143	119		129	157	61	43
8	177						117	63
9								179
10								
11								
12								

Table 3

Students graphed both sets of data, and most could see right away that the decay experiment produces a decreasing exponential curve. They could also give an explanation, "About half of the coins disappear each time." The students also guessed, graphed, and convinced themselves that the second experiment models exponential growth. Eventually, several students saw symmetry in the two sets of data. The columns in the growth table look almost identical to the columns in the decay table turned upside down. Students should then be able to make a conjecture that the growth rate was 2 and to give a reason based on the growth model. "If we start with n coins in any generation where n is the total number of coins in all groups, approximately half of the coins will turn up tails. No additional coins are added from these particular coin tosses. These tosses contribute $0.5n$ to the next generation. The other half landed heads. Each of these heads contributes three coins to the next generation—itself plus the two added. Thus, the total contribution from the heads coin tosses is $0.5 \cdot 3 \cdot n$." After making this discovery, the students should be able to refer directly to different models for growth and decay and correctly give the rate.

In the next issue you will see how to expand on the concepts presented in these activities to model the dynamics of population growth and decay.