The Exponential Function and the Dynamics of Population, Part 2

by William Yslas Velez and Catherine M. Yslas
University of Arizona

The Southwest Regional Institute in the Mathematical Sciences (SWRIMS), based at the University of Arizona, was a National Science Foundation project whose purpose was to shed light on how university researchers can become involved in education. SWRIMS believes that mathematics can be made meaningful and germane to K–14 instructors and students of all levels. One outcome of SWRIMS was a series of four activities related to the dynamics of population growth. The first two activities were included in the December 1998 *ComMuniCator*. The third and fourth are given below.

Activity 3: Population, Plenty, and Poverty

The goal for the third set of activities is to help students develop ideas about the purposes of mathematical modeling and, at the same time, gain some understanding of the limitations of these models. A good context for students to achieve this goal is to have them try to model human population dynamics. If populations are to be modeled, it is not enough to look solely at the total population. Instead, different segments of the population, such as children and adults, and their interaction need to be studied. Below is a summary of an age-structured population model of children and adults. In considering this segmented population, the birth, death, and maturation rates need to be decided. The students then have the option of testing the dynamics of the population change while constructing a mathematical model.

Students can relate the results of the paper folding investigation in Activity 1 to population dynamics. For example, students can look at the social structure of Kenya. Kenya's population has doubled five times in this century. If a piece of land in 1900 is represented by a whole sheet of paper, then today's average farmer has to make do with the reduced relative amount of land represented by the paper after five folds. Will Kenya be able to face a situation that can be made symbolically by making a sixth paper fold? Another fascinating example is modeling the population dynamics of China. With the Chinese government only allowing one child per family, students try to answer the question, "How do the Chinese know for sure that the population will not collapse someday?"

Propose the following scenario to students—a 20% birth rate, a 30% maturation rate, and a 20% death rate—and ask them to model a population using a five-year census interval. In this model no children die. The class can choose three officials to direct the dynamics. The "stork" counts the number of adults and determines the number of new babies by the time of the next census using the equation

births = birth rate · adult population.

The "escort" counts the number of children and computes the number of children who have matured and become adults by the next census.

maturing children = new adults =
maturation rate · child population

The "grim reaper" culls the adult population using the equation

deaths = death rate · adult population.

After the calculations are complete, the students can act out the population changes and record the results in a table. Then the students are asked to write two new equations to describe the population dynamics.

children in the next census =
children + births – maturing children

adults in the next census =
adults – deaths + maturing children

One of our classes developed the table below as their country's census data.

census	children	adults	births	maturing children	deaths
0	5	20	4	2	4
1	7	18	4	2	4
2	9	16	3	3	3
3	9	16	3	3	3
4	9	16			

Let X(t) = *number of children at census t,*
and Y(t) = *number of adults at census t.*

Recall

children in next census =
children + births – maturing children

$$X(t+1) = X(t) + 0.2 \cdot \textit{adults} - 0.3 \cdot \textit{children} = 0.7X(t) + 0.2Y(t)$$

adults in next census =
adults – deaths + maturing children

$$Y(t+1) = Y(t) - 0.2 \cdot \textit{adults} + 0.3 \cdot \textit{children} = 0.3X(t) + 0.8Y(t)$$

Adding the two equations above, the students discover why the total population becomes stable.

$$\textit{total population} = X(t+1) + Y(t+1) = X(t) + Y(t)$$

If the population is stable, then

$$X(t) = X(t+1) = (2/3)Y(t),$$

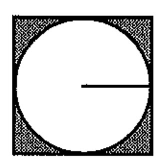
which agrees with their previous observations.

These four activities demonstrate the use of the exponential function in a large variety of contexts—paper folding, walking, the growth of bacterial cultures, coin tossing, sizes of generations, and human populations. Many other possibilities also exist and might be more relevant examples for different mathematics classes. For example, radioactive decay follows an exponential decay. Another example is dropping a ball and recording the height of successive bounces. A third example is based on Newton's law of cooling (or heating). According to this law, if an object at a certain temperature is placed in an environment at a different temperature, then the difference between the temperature of the object and the environment decays exponentially with time. By investigating real-world phenomena, students can develop a much better understanding of the exponential function.

Below is a program for the TI-82 and TI-83 for use with these activities:

```
PROGRAM:POPDYN
ClrHome
ClrList (L/,L¤,L‹,L›,Lfi,Lfl)
Input "BIRTH RATE? ",B
Input "MATURATION RATE? ",M
Disp "DEATH RATE"
Input "CHILDREN? ",X
Input "ADULTS? ",D
Disp "INITIAL"
Disp "POPULATION"
Input "CHILDREN? ",C
Input "ADULTS? ",A
0 L/(1)
C L¤(1)
A L‹(1)
A+C L›(1)
C/A Lfi(1)
0 Lfl(1)
Disp "FINAL GENERATION"
Input F
For(G,1,F,1)
G L/(G+1)
B*L‹(G)+(1-M-X)*L¤(G) L¤(G+1)
(1-D)*L‹(G)+M*L¤(G) L‹(G+1)
L¤(G+1)+L‹(G+1) L›(G+1)
L¤(G+1)/L‹(G+1) Lfi(G+1)
L›(G+1)/L›(G) Lfl(G+1)
End
```

Circle/Square Ratio Revisited in Volume



by Gwen Hogin
Turlock Jr. High School

In the June 1991 issue of the *ComMuniCator*, I wrote an article that used the ratio of the area of a circle to the area of a square with sides tangent to the circle. This activity helped students understand the concept of ratios and encouraged them to explore an alternative way to calculate the area of a circle. You can readily show this ratio using a circle with a radius of 1 unit. The area of the circle would then be 1 x 1 x 3.14 square units. The square in which it is inscribed would have a side of 2 units and an area of 4 square units. Thus, 3.14 divided by 4 is equal to 0.785. Any circle's diameter can be squared and multiplied by 0.785 to determine its area.

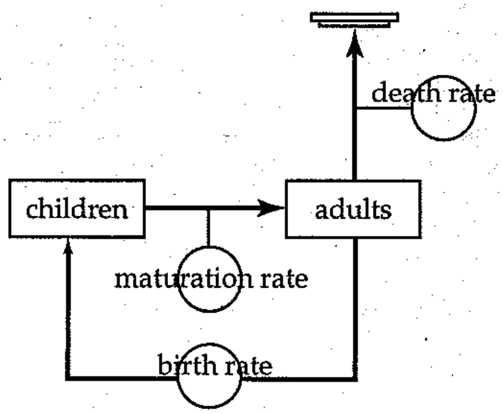
This same concept can be used to calculate the volume of a cylinder. One can square the diameter of the base and multiply by 0.785 and then the height to determine the volume.

The students were amazed that the population of 9 children and 16 adults is stable. Now students are ready for the fourth activity in which they investigate this stability and learn what happens when the birth, maturation, and death rates are changed.

Activity 4: Models of Populations

In this final set of activities, students use computers or a graphing calculator to calculate the dynamics of the age-structured population to learn about stable population distributions and stable population growth rates. This activity demonstrates that the exponential function can appear as a consequence of modeling, and the growth rate cannot be discovered simply from the parameters of the model.

First, students create a stock-flow diagram of their population model, which they use to help them calculate population sizes. They could even move beans around the diagram to represent members of an evolving population.



Ask students to begin with the same values they used for the birth, maturation, and death rates as in the third activity. This choice allows them to begin by investigating what they suspect is a stable population. Our class started by imagining a large space ship with 40,000 adults and no children headed for Alpha Centuri, the nearest star to our solar system. Using their computer or calculator program or beans, the students watched the population evolve over time (*Figure 1*).

Figure 1

census	children	adults	census	children	adults
0	0	40000	10	15984	24016
1	8000	32000	11	15992	24008
2	12000	28000	12	15996	24004
3	14000	26000	13	15998	24002
4	15000	25000	14	15999	24001
5	15500	24500	15	15999	24001
6	15750	24350	16	16000	24000
7	15875	24125	17	16000	24000
8	15938	24062	18	16000	24000
9	15969	24031	19	16000	24000
			20	16000	24000

The students graphed their data and made several observations that included a weakness in the model, which they rectified. In constructing the mathematical model, the instructors never wrote ab^t for any values of a or b. A few students noticed that the exponential function could be seen from the data in the table—the increase in children in any census is always half what it was in the previous census. Other students noticed that the exponential function was represented by the graph. With help from the instructor, the students observed that the equation for the graph was

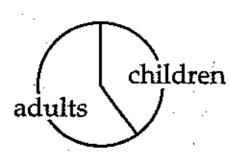
$$\textit{children population at census } k = 16{,}000 - 16{,}000 \cdot (1/2)^k$$

The students were then issued a new challenge—if they knew the initial population, could they predict the stable population? The students found that the stable population depended only on the size of the population. The groups picked different population sizes and reported the stable population.

Initial Population	Stable Population	
Total	Children	Adults
40000	16000	24000
10000	4000	6000
80000	32000	48000
5000	2000	3000

The students did not see a pattern in the table until they drew pie charts of the stable populations. The charts are all the same: 40% are children and 60% are adults. In other words,

$$\textit{children population} = (2/3) \cdot \textit{adult population}$$



When the students turn the above relationship into equations, they make an amazing mathematical discovery.

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