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Living GRAPHS

GRAPHS ARE ALL AROUND US. Open a newspaper, and you will find graphs representing myriad items of information, ranging from descriptions of the past decade's crime rate to the average price of one week's groceries at local supermarkets. A graph serves as a focusing agent; it makes a visual point. You may not remember the text, but the writer hopes that you will remember the accompanying graph and, for example, purchase the Safe Security System because the crime-rate graph is rising dramatically or shop at the Great Food Basket because the graph shows the lowest prices there. By looking at the peaks and valleys of a graph, a scientist reaches conclusions or formulates hypotheses. The slope of a portion of a graph causes an engineer to make a decision. A graph is data made visual. The visual language associated with graphs—*rising, lowest point, peak, steepness*—gives them, and the data from which they were created, lives of their own.

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In this article, we discuss some ideas and techniques for teaching and learning this language and for giving life to graphs in middle school classrooms. In algebra, students typically construct graphs of functions represented by symbolic formulas. Because the formula comes first, the graph becomes a secondary object. Moreover, if the formula is not tied to a real problem, the graph becomes even more remote and abstract. We would like to enlarge the students' role in creating the notion of a graph. We would like to make graphs *real* and elevate the visual thinking that goes with graphs to the same level as symbolic thinking. In calculus, both these skills should be second nature.

In rethinking how to present graphing in middle school, we decided that we should and could present the topic without formulas and algebra, an area in which most middle school students have not had much experience. We believed that a number of goals could be accomplished using graphing, including encouraging students to think visually about functions and to solve problems using graphs, without algebra as either extra baggage or a crutch. Because most graphs arise without formulas, having students construct graphs without formulas would be a natural approach. No formula was used to create the crime-rate figures mentioned previously, for example; nor was a formula used to plug a particular store into this week's average grocery prices.

Our approach would offer a number of interesting examples to present to students, and technology would support us in our efforts. Data collection and point-by-point graphing are labor-intensive activities that tend to obscure the important aspects of the task. Technology would help us bypass this mundane work and leave more time for analyzing the visual features of the graph, using these aspects to understand the function being graphed, and solving problems associated with the function. Because we wanted graphing to become second nature, we also wanted the students to touch, feel, and act out the ideas being represented. Visual clues should be associated with actions; students should feel and experience such concepts as increasing and decreasing, concave up and concave down, and maximum and minimum.

We introduced our formula-free, action-oriented graphing activities in Making Everybody Count, a teacher-enhancement program funded by the National Science Foundation for middle school teachers from Arizona. We also tried the activities in a iiddle school classroom. The following paragraphs recount what took place in Sonya Jackson's sixth-grade class at Gridley Middle School, Tucson, Arizona.

Motion along a Straight Line

A SPACE SCIENTIST MUST CORRECTLY DETERMINE the path that a spaceship takes to land on the moon. A baseball player must accurately judge the path of a fly ball to catch it. Determining the motion of an object is of fundamental importance in scientific inquiry, as well as for survival, as anyone who drives our freeways knows. Some motions are too complicated for middle schoolers to explore, but motion along a straight line is accessible to them, and this activity focuses on graphs that capture such motion.

Imagine a string tightly strung between two points and a bug traveling along it. The path traced is motion along a straight line. To study the bug's motion, we collect data—each data point consists of a pair of numbers that tells the position of the bug along the line and the instant in time that the bug was at that position. To understand these data, we traditionally graph them so that the horizontal axis represents time and the vertical axis, position. A two-dimensional picture is used to describe one-dimensional motion! Studying these graphs takes up much of calculus.

Motion along a straight line includes a variety of real-world situations, for example, an elevator going up and down; a ball being tossed straight up into the air; an object being dropped from a tall building; the end of a delicate spring, such as the spring that controls a seismograph, being set in motion; the level of water in a bathtub as it is being filled or emptied; and so on. Such graphs also have the potential to encourage students' participation and pique students' interest in creating and interpreting them. Certainly, this potential met an important criterion for us: if we wanted to devise activities in which graphs come alive for students, then the students must participate in every aspect of the activities.

Spaceships

Although we know that graphs of straight-line motion are important in the mathematical world, how could we promote middle school students' interest in straight-line motion? We started with spaceships. We said to the students, "We—all of us—are sending a spaceship to the moon. What factors are important for us to know ahead of time about the trip?" Students came up with numerous answers, such as the distance from the earth to the moon, the length of time the trip would take, the speed at which we would travel, the amount of food we would have to take, and so on. Eventually, their responses led to a discussion of the motion of the earth and the moon and the fact that the moon travels around the earth in a more-or-less circular orbit. We had students model the orbit. One student stood up and played the stationary earth; another played the moon. To get the spaceship in the picture, we stretched a piece of yarn to represent the path that the spaceship would take from the earth to the moon. A small paper clamp threaded on the yarn represented our spaceship, and attached to the clamp was a small plastic bag—our command module—with a figure inside—our astronaut! A student moved the clamp along the yarn from the earth to the moon in a straight-line path.

Motion along a straight line includes a variety of real-world situations

Students soon realized that the motion of the moon had not been taken into account. So far, both the moon and the earth had been stationary. When the moon is in motion, however, the end of the stationary straight-line path no longer touches the moon, which it must do if the spaceship is to reach the moon. In addition, the path would have to touch the moon at the exact instant that the astronaut on the yarn was also touching the moon. Everyone realized that the path is very complicated. In particular, the spaceship does not travel in a straight line.

Earthships

Straight-line spaceship travel seemed like an exciting area to explore. We decided to investigate the motion of "earthships," which travel along a straight line. Before the class started, we stretched twine between two chairs at the front of the room to simulate the line of motion for our earthship. Along the twine, we hung cards spaced about a foot apart and labeled 1, 2, 3, 4, 5, 6, 7 in succession from left to right (see **fig. 1**). To add complexity and compensate for the fact that real space travel was out of the question, we allowed the earthship to travel back and forth along the line in any way that the students chose.

We selected a student to be the earthship and asked that person to "take a trip along the line," that is, walk back and forth a few inches behind and parallel to the twine. We asked the class what sort of record should be made to enable another earthship to reproduce the trip at a later time. The students suggested using the numbers on the line, as well as the number of steps taken forward and backward. After some discussion, we decided to test the students' method. We chose one student as a "virtual" earthship and asked that individual to leave the room while the "real" earthship took a trip along the line and the observing students kept a record. The virtual earthship returned to the room to replicate the motion of the real earthship, relying on the comments of the students who had remained in the classroom. Students called out suggestions: "This way!" "No, that way!" Chaos reigned for a bit. Eventually, the students were able to guide the virtual earthship to take a trip that more or less resembled the original trip. The class was ready, however, for a more systematic method of recording measurements during the trip and a better way of replicating the motion.

Living graph

We wanted the students to understand the idea that two measurements would be helpful to describe motion on a straight line. These measurements were elapsed time and position of the earthship at the time elapsed. We would take measurements at equal time intervals. As the earthship took its trip, one of the facilitators would clap about once every second and call out, "clap 1, clap 2," and so on. At every clap, the position of the earthship, as determined by the markers placed along the line, would be recorded. We would clap ten times and record ten positions. We chose ten student-recorders to sit about three feet from the line, parallel to and facing it to observe the earthship and the markers (see **fig. 2**). Each recorder was assigned a number from 1 through 10, corresponding to clap 1 through clap 10, and would be responsible for recording the position at just one clap. We told the students that they should record the position of the spaceship when it was their turn and their clap number came up. We also tapped each student on the shoulder when it was his or her turn. To the students, this system had many of the same elements as their earlier, less organized, method. The new element was the careful use of time.

After practicing this routine, students questioned how to obtain their estimates of position. What is the position when the earthship is between markers? The students resolved this issue quickly and agreed to use one-place decimal fractions as estimates. Despite this agreement, one student produced a measurement of 4.12. We practiced the process again until the students were comfortable with it. Finally, we ran the process in earnest. The ten students recorded and memorized their measurements of position.

On the opposite side of the room, we had stretched twine between two additional chairs. The twine was about two feet from, and parallel to, the floor and would serve as a timeline. Along this twine, we hung ten equally spaced cards labeled clap 1 through clap 10 (see **fig. 3**). We chose ten additional students, one to stand behind each of the ten cards and hold a yardstick. At the top of each yardstick, a hole had been drilled and a single long piece of yarn had been threaded through all the holes. The yarn fell loosely between the yardsticks.

We asked the recorders to go to the "clap line" and give their position measurements to the students holding yardsticks. The recorder of the earthship's position at clap 1 would give his or her position measurement to the student holding the yardstick and standing behind the clap 1 label, and so on for all the claps. Each pair of students then maneuvered the yardstick so that the hole at its end was a distance above the clap line equal to the recorded measurement. The yardsticks were only three feet long, yet some measurements were as much as six feet. Students helped one another position the yardsticks correctly. The facilitators also made sure that everyone understood the instructions, and we tightened the yarn joining the yardsticks. The rest of the class stood back and looked at the results of our labor. We had created a "living" graph to represent the earthship's trip (see **fig. 4**).

Testing understanding

Students walked along this living graph, with their hands gently touching the yarn joining the holes in the yardsticks. As they felt their hands going up and down, we had them say "up" or "down." This exercise gave them a rough feeling for what the earthship does on its trip: as the hand moves up, the earthship moves forward; as it moves down, the earthship moves backward.

We tested this tactile association by selecting another student to be the earthship. We asked her to replicate the motion of the first earthship using the living graph as a guide. She readily associated motion with the position of the yardsticks. She knew when to move forward and backward and how far to move. We then devised another test. We asked the students in the graph to change the positions of their yardsticks, then asked a new earthship to move according to the new graph. The earthship student knew exactly what to do.

Interpolation

Next we asked the students to return the yardsticks to the positions for the original earthship graph. We asked what the position of the earthship was when it was halfway between claps 3 and 4, that is, at clap 3.5. The students estimated that the position should be halfway between the positions at claps 3 and 4. We asked them where that position would be located on the graph. The students had us pull the yarn taut to make straight lines between the tops of the yardsticks. A student moved her hand along the tight yarn until it was halfway between clap 3 and clap 4 and, without any calculation, made her estimate. The position of her hand on the yarn—indicating the vertical distance from the hand to the clap line—was the estimate! We asked the students whether they were sure about this estimate, that is, did the taut yarn really indicate the position of the earthship between claps? The students admitted that maybe the position was not completely accurate, but using the straight lines was a good tool for estimating.

We let the yarn go slack and used the new position of the yarn to make an estimate—the distance from the clap line to the yarn—which the students agreed was just as good as the first. "How could you be really sure where the earthship was at clap 3.5?" we asked. "Take more measurements," they answered. "If we had twice the number of measurements, then we would be able to answer the question exactly." Then we asked where the spaceship would be at clap 3.1. "Take even more measurements," they responded. How many measurements in all? One hundred? That exercise would require one hundred recorders, one hundred yardsticks, and one hundred students to hold the yardsticks. "No way!" the students cried.

Technology to the rescue

We then told the students that we had a piece of hardware that could do exactly what we were doing, but instead of ten measurements, it could take a hundred measurements or a thousand measurements or more in the same period of time. This device is the Texas Instruments Calculator-Based Laboratory (TI-CBL), which has a distance probe connected to a calculator and the calculator connected to an overhead-display unit. We showed the students the CBL and explained that the probe would record the positions of the earthship every 1/10 of a second for ten sec-

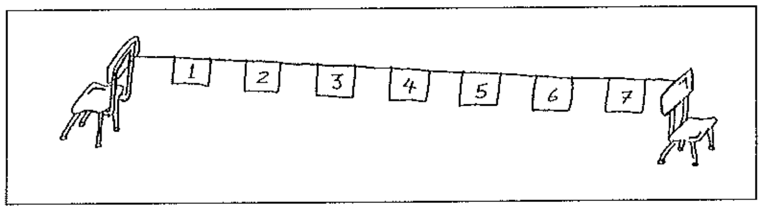


Fig. 1 **Line of motion for earthship travel**

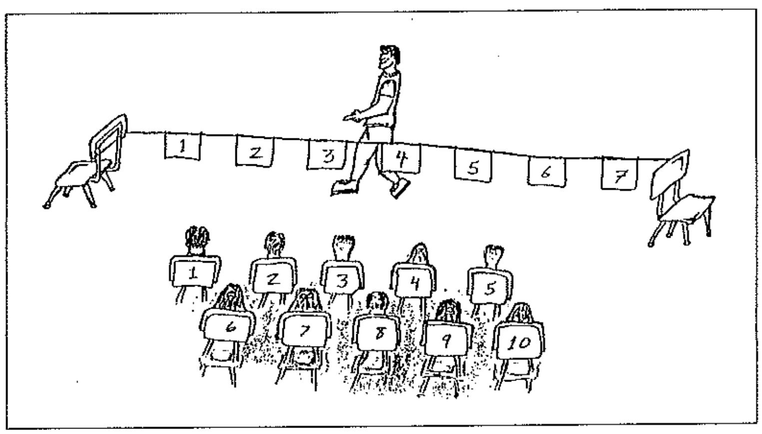


Fig. 2 **Students assigned to record earthship position**

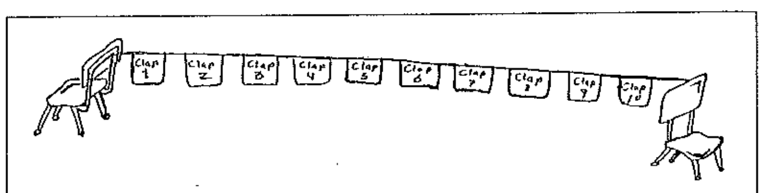


Fig. 3 **Timeline configuration**

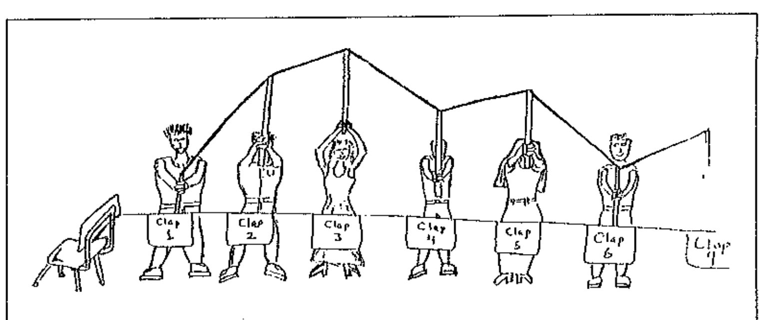
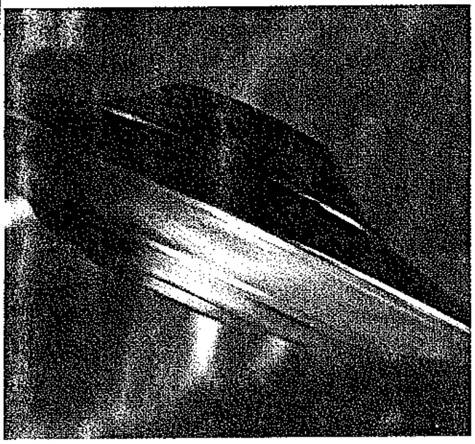


Fig. 4 **Our "living" graph**

onds. A few seconds later, the CBL would produce the graph on an overhead projector. This graph would be similar to the one that we had produced earlier with yardsticks and fewer measurements. With the students, we performed the CBL experiment with the earthship, first making one observation every second, then one every tenth of a second, then one every thousandth of a second.



The students were amazed at the performance of the technology

Representing the earthship's trip with a yardstick graph yields a discrete set of points, corresponding to the tips of the yardsticks. The yarn that we had used to connect the tips of the yardsticks was an illusion. We had determined no data points for in-between times, but the yarn suggested the possibility of more data points. As the number of observations increases, the data begin to resemble a continuous curve. When the CBL is used with one hundred measurements, the result looks very much like a curve. We asked the students to take turns walking in front of the CBL distance probe. The students were entranced with, and amazed at, the performance of the technology.

More Activities with the CBL and Beyond

THE CBL IS A WONDERFUL TOOL TO EXPLORE THE connection between motion and graphs. **Table 1** lists the actions that students can take and describes the corresponding graphs that the CBL will display. Keep in mind that the CBL motion detector will detect an object only when it is between two and five feet away from the sensor. To ensure that the graph displayed looks like a curve, set the CBL to take a measurement about once every hundredth of a second.

TABLE 1
Student Actions and Their Resulting CBL Graphs

STUDENT ACTIONS	GRAPH DISPLAYED BY CBL
Stand still; zero speed	Horizontal line
Walk away from sensor	Graph rises from left to right
Walk toward sensor	Graph descends from left to right
Change directions, that is, turn around	Graph shows a peak or valley
Walk at a constant speed	Graph is a straight line
Walk away from sensor; start slowly, and gradually speed up	Graph curves up (is concave up)
Walk toward sensor; start slowly, and gradually speed up	Graph curves down (is concave down)

Another activity is to have one student try to replicate the motion of another using the CBL. Have a student walk in any way he or she chooses. Once the graph has been projected onto the screen, place a clear transparency over the overhead display unit and trace the graph on the transparency. Keeping the traced graph on the overhead transparency, have another student take over as walker. Instruct that student to walk so that his or her graph will be as close as possible to the first walker's graph. Students may need several practice runs before they can begin to approximate the first walker's actions. The audience cannot help but give advice: "Go faster, go slower, change directions!"

After some work with this activity, you can ask students to predict how the graphs will look if certain motions take place. For example, if two people move with different constant speeds, which graph corresponds to the greater of the two speeds? Conversely, you can give students graphs and ask them to describe what the corresponding motions should be. For example, what can you say about the motions for the two straight-line graphs shown in **figure 5?**

Of course, some graphs do not correspond to CBL actions. Two examples are shown in **figure 6.** Ask the students to what motions these graphs correspond. When the students are acting out the living graph with yardsticks, you might suggest that some of them place the tips of the yardsticks *below* the timeline. Then ask them to state the motion to which that configuration corresponds.

Eventually, students will not need the CBL at all to make connections between motion along a straight line and its graph. At this point, you can have students draw graphs that correspond to stories or, conversely, write stories that correspond to graphs. Of course, the stories must be about trips on a straight line. Students could walk back and forth in front of their houses, walk to the store directly down the street, or jump up and down on a trampoline. They could then write stories about these activities and draw the associated graphs. **Figure 7** shows an example of a story and a graph from Wendy Diskin's sixth-grade class at Mayer Elementary School in Mayer, Arizona. The students could also trade stories, try to draw the graphs described in one another's stories, and compare the results. Finally, you could give your students graphs and ask them to construct stories about trips that would yield those results.

Summary

IN ALL THE ACTIVITIES WE HAVE DESCRIBED, THE students are physically involved. When they use the yardsticks, they are actually a part of the graph itself. When they use the CBL, their actions are directly connected with the output that they see. Moreover, the CBL graph is dynamic—just as students' trip taking is action packed—because they see the graph evolving as they walk. When the yardsticks and graphs are removed and students begin writing stories and drawing the graphs that go with them, the physical involvement is replaced by an emotional one. Each story is related to an individual's own action; each graph is connected with a student's personal experience. These graphs belong to the students. At the end of these activities, the students have internalized a visual language and developed habits of mind that will serve them well in high school, in college, and in careers that use graphs as tools of description, analysis, and problem solving.

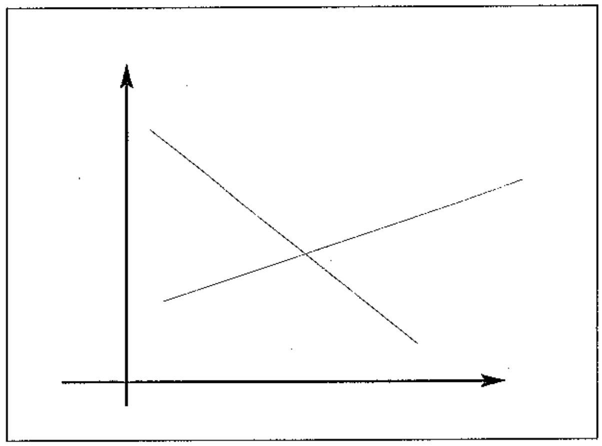


Fig. 5 Predict the motions that yielded these CBL graphs.

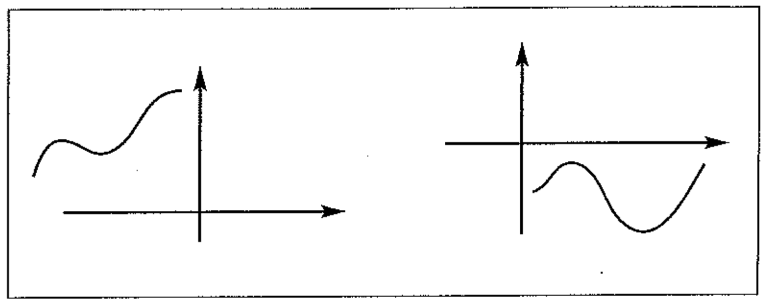


Fig. 6 Graphs that do not correspond to CBL actions

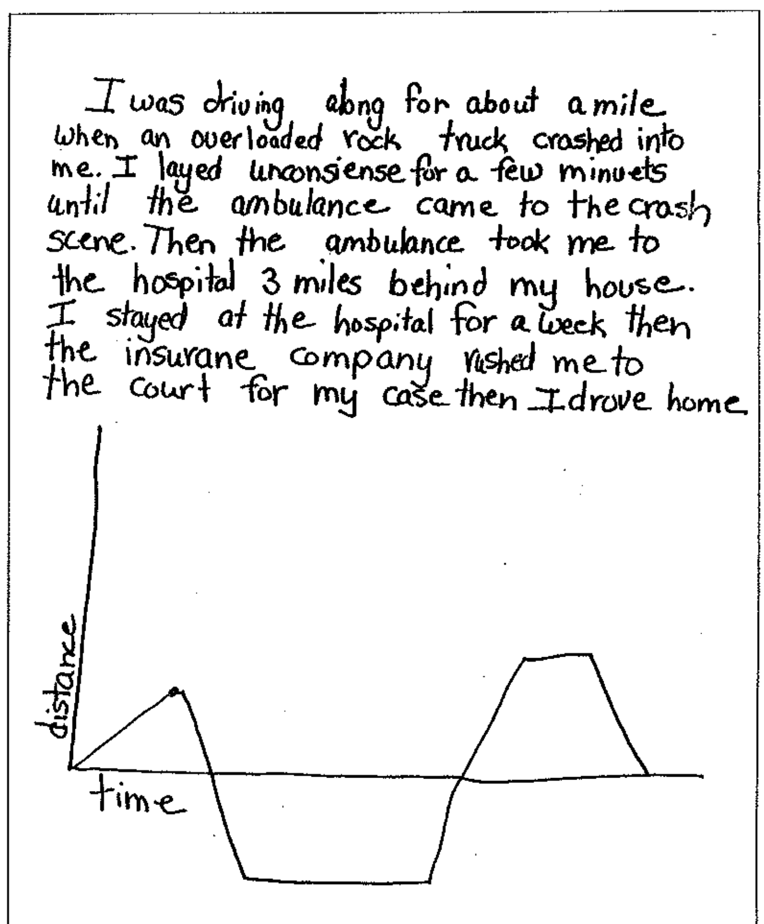


Fig. 7 A student's story and graph about a straight-line trip

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