Chapters 1-2-4: Ordinary Differential Equations

Sections 1.1, 1.7, 2.2, 2.6, 2.7, 4.2 & 4.3

1. Ordinary differential equations

- An ordinary differential equation of order n is an equation of the form

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right). \tag{1}$$

- A solution to this differential equation is an n-times differentiable function $y(x)$ which satisfies (1).
- **Example:** Consider the differential equation

$$y'' - 2y' + y = 0.$$

  - What is the order of this equation?
  - Are $y_1(x) = e^x$ and $y_2(x) = x\,e^x$ solutions of this differential equation?
  - Are $y_1(x)$ and $y_2(x)$ linearly independent?

Initial and boundary conditions

- An initial condition is the prescription of the values of y and of its $(n-1)$st derivatives at a point x_0,

$$y(x_0) = y_0, \ \frac{dy}{dx}(x_0) = y_1, \ldots \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}, \qquad (2)$$

  where y_0, y_1, ... y_{n-1} are given numbers.

- Boundary conditions prescribe the values of linear combinations of y and its derivatives for two different values of x.

- In MATH 254, you saw various methods to solve ordinary differential equations. Recall that initial or boundary conditions should be imposed after the general solution of a differential equation has been found.

2. Existence and uniqueness of solutions

- Equation (1) may be written as a first-order system

$$\frac{dY}{dx} = F(x, Y) \qquad (3)$$

  by setting $Y = \left[y, \frac{dy}{dx}, \frac{d^2y}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right]^T$.

- Existence and uniqueness of solutions: if F in (3) is continuously differentiable in the rectangle

$$R = \{(x, Y), \ |x - x_0| < a, \ ||Y - Y_0|| < b, \ a, b > 0\},$$

  then the initial value problem

$$\frac{dY}{dx} = F(x, Y), \qquad Y(x_0) = Y_0,$$

  has a solution in a neighborhood of (x_0, Y_0). Moreover, this solution is unique.

Existence and uniqueness of solutions (continued)

- **Examples:**
  - Does the initial value problem
  $$y'' - 2y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$
  have a solution near $x = 0,\ y = 1,\ y' = 0$? If so, is it unique?
  - Does the initial value problem
  $$y' = \sqrt{y}, \qquad y(0) = y_0$$
  have a unique solution for all values of y_0?
  - Does the initial value problem
  $$y' = y^2, \qquad y(1) = 1$$
  have a solution near $x = 1,\ y = 1$? Does this solution exist for all values of x?

Existence and uniqueness for linear systems

- Consider a linear system of the form
$$\frac{dY}{dx} = A(x)Y + B(x),$$
where Y and $B(x)$ are $n \times 1$ column vectors, and $A(x)$ is an $n \times n$ matrix whose entries may depend on x.

- Existence and uniqueness of solutions: If the entries of the matrix $A(x)$ and of the vector $B(x)$ are continuous on some open interval I containing x_0, then the initial value problem
$$\frac{dY}{dx} = A(x)Y + B(x), \qquad Y(x_0) = Y_0$$
has a unique solution on I.

Existence and uniqueness for linear systems (continued)

- **Examples:**
  - Apply the above theorem to the initial value problem

$$y'' - 2y' + y = 3x, \qquad y(0) = 1, \qquad y'(0) = 0$$

  - Does the initial value problem

$$\begin{aligned} &y^{(4)} - x^3 y'' + 3y = 0, \\ &y(0) = 1,\ y'(0) = 1,\ y''(0) = 0,\ y^{(3)}(0) = 0 \end{aligned}$$

    have a unique solution on the interval $[-1, 1]$?

3. Linear differential equations and systems

- The general solution of a homogeneous linear equation of order n is a linear combination of n linearly independent solutions.
- As a consequence, if we have a method to find n linearly independent solutions, then we know the general solution.
- In MATH 254, you saw methods to find linearly independent solutions of homogeneous linear ordinary differential equations with constant coefficients.
- This includes linear equations of the form $ay'' + by' + cy = 0$, and linear systems of the form $\frac{dY}{dx} = A\,Y$, where A is an $n \times n$ constant matrix and $Y(x)$ is a column vector in $\mathbb{R}^n$.

Linear differential equations and systems (continued)

- A set $\{y_1(x), y_2(x), \cdots, y_n(x)\}$ of n functions is linearly independent if its Wronskian is different from zero.
- Similarly, a set of n vectors $\{Y_1(x), Y_2(x), \cdots, Y_n(x)\}$ in $\mathbb{R}^n$ is linearly independent if its Wronskian is different from zero.
- The Wronskian of n functions $y_1(x),\ y_2(x),\ \cdots,\ y_n(x)$ is given by

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ {y_1}'' & {y_2}'' & \cdots & {y_n}'' \\ \vdots & \vdots & \ddots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix}$$

Ordinary differential equations
Linear differential equations and systems
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General facts
Homogeneous linear equations with constant coefficients
Homogeneous linear systems with constant coefficients

Linear differential equations and systems (continued)

- The Wronskian of n vectors $Y_1(x),\ Y_2(x),\ \cdots,\ Y_n(x)$ in $\mathbb{R}^n$ is given by

$$W(Y_1, Y_2, \cdots, Y_n) = \det\left([Y_1\ \ Y_2\ \ \cdots\ \ Y_n]\right),$$

where $[Y_1\ \ Y_2\ \ \cdots\ \ Y_n]$ denotes the $n \times n$ matrix whose columns are $Y_1(x),\ Y_2(x),\ \cdots,\ Y_n(x)$.
- Finding n linearly independent solutions to a homogeneous linear differential equation or system of order n, is equivalent to finding a basis for the set of solutions.
- The next two slides summarize how to find linearly independent solutions in two particular cases.

Chapters 1-2-4: Ordinary Differential Equations

Homogeneous linear equations with constant coefficients

To find the general solution to an ordinary differential equation of the form $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$, proceed as follows.

1. Find the characteristic equation, $a\lambda^2 + b\lambda + c = 0$ and solve for the roots λ_1 and λ_2.
2. If $b^2 - 4ac > 0$, then the two roots are real and the general solution is $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
3. If $b^2 - 4ac < 0$ the two roots are complex conjugate of one another and the general solution is of the form $y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$, where $\alpha = \Re e(\lambda_1) = \frac{-b}{2a}$, and $\beta = \Im m(\lambda_1) = \frac{\sqrt{4ac-b^2}}{2a}$.
4. If $b^2 - 4ac = 0$, then there is a double root $\lambda = -\frac{b}{2a}$, and the general solution is $y = (C_1 + C_2 x)\, e^{\lambda x}$.

Homogeneous linear systems with constant coefficients

To find the general solution of the linear system $\dfrac{dY}{dx} = A\,Y$, where A is an $n \times n$ matrix with constant coefficients, proceed as follows.

1. Find the eigenvalues and eigenvectors of A.
2. If the matrix has n linearly independent eigenvectors $U_1, U_2, \cdots, U_n$, associated with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the general solution is
$$Y = C_1 U_1 e^{\lambda_1 x} + C_2 U_2 e^{\lambda_2 x} + \cdots + C_n U_n e^{\lambda_n x},$$
where the eigenvalues λ_i may not be distinct from one another, and the C_i's, λ_i's and U_i's may be complex.

If A has real coefficients, then the eigenvalues of A are either real or come in complex conjugate pairs. If $\lambda_i = \overline{\lambda_j}$, then the corresponding eigenvectors U_i and U_j are also complex conjugate of one another.

4. Nonhomogeneous linear equations and systems

- The general solution y to a non-homogeneous linear equation of order n is of the form
$$y(x) = y_h(x) + y_p(x),$$
where $y_h(x)$ is the general solution to the corresponding homogeneous equation and $y_p(x)$ is a particular solution to the non-homogeneous equation.
- Similarly, the general solution Y to a linear system of equations $\frac{dY}{dx} = A(x)Y + B(x)$ is of the form
$$Y(x) = Y_h(x) + Y_p(x),$$
where $Y_h(x)$ is the general solution to the homogeneous system $\frac{dY}{dx} = A(x)Y$ and $Y_p(x)$ is a particular solution to the non-homogeneous system.

Nonhomogeneous linear equations and systems (continued)

- In MATH 254, you saw methods to find particular solutions to non-homogeneous linear equations and systems of equations.
- You should review these methods and make sure you know how to apply them.