Chapter 13: Complex Numbers

Sections 13.3 & 13.4

1. Function of a complex variable

- A (single-valued) function f of a complex variable z is such that for every z in the domain of definition $\mathcal{D}$ of f, there is a unique complex number w such that

$$w = f(z).$$

- The real and imaginary parts of f, often denoted by u and v, are such that

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy, \quad x, y \in \mathbb{R}, \quad f(z) \in \mathbb{C},$$

with $u(x, y) \in \mathbb{R}$ and $v(x, y) \in \mathbb{R}$.

Function of a complex variable (continued)

- **Examples:**
  - $f(z) = z$ is such that $u(x, y) = x$ and $v(x, y) = y$.
  - Find the real and imaginary parts of $f(z) = \bar{z}$.
  - $f(z) = \dfrac{1}{\bar{z}}$ is defined for all $z \neq 0$ and is such that

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = \frac{y}{x^2 + y^2}.$$

2. Limits and continuity

- An open neighborhood of the point $z_0 \in \mathbb{C}$ is a set of points $z \in \mathbb{C}$ such that

$$|z - z_0| < \epsilon, \qquad \text{for some } \epsilon > 0.$$

- Let f be a function of a complex variable z, defined in a neighborhood of $z = z_0$, except maybe at $z = z_0$.
- We say that f has the limit w_0 as z goes to z_0, i.e. that

$$\lim_{z \longrightarrow z_0} f(z) = w_0,$$

if for every $\epsilon > 0$, one can find $\delta > 0$, such that for all $z \in \mathcal{D}$,

$$|z - z_0| < \delta \Longrightarrow |f(z) - w_0| < \epsilon.$$

- **Example:** $\displaystyle\lim_{z \rightarrow i} \frac{z^2 + 1}{z - i} = 2i.$

Continuity

- The function f is continuous at $z = z_0$ if f is defined in a neighborhood of z_0 (including at $z = z_0$), and

$$\lim_{z \longrightarrow z_0} f(z) = f(z_0).$$

- If $f(z)$ is continuous at $z = z_0$, so is $\overline{f(z)}$. Therefore, if f is continuous at $z = z_0$, so are $\Re e(f)$, $\Im m(f)$, and $|f|^2$.
- Conversely, if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$, is continuous at $z_0 = x_0 + iy_0$.
- **Example:** Is the function such that $f(z) = \Im m(z^2)/|z|^2$ for $z \neq 0$ and $f(0) = 0$, continuous at $z = 0$?

3. Differentiability

- Assume that f is defined in a neighborhood of $z = z_0$. The derivative of the function f at $z = z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

  assuming that this limit exists.

- If f has a derivative at $z = z_0$, we say that f is differentiable at $z = z_0$.

- **Examples:**
  - $f(z) = \bar{z}$ is continuous but not differentiable at $z = 0$.
  - $f(z) = z^3$ is differentiable at any $z \in \mathbb{C}$ and $f'(z) = 3z^2$.

Rules for continuity, limits and differentiation

- To find the limit or derivative of a function $f(z)$, proceed as you would do for a function of a real variable.
- **Examples:**
  - $f'\left(\dfrac{1}{z}\right) = -\dfrac{1}{z^2}$.
  - $\dfrac{d}{dz} z^n = n\, z^{n-1}, \qquad n \in \mathbb{N}$.
  - Find $\displaystyle\lim_{z \to -i} \left(z + \frac{1}{z}\right)$.

Rules for continuity, limits and differentiation (continued)

- Properties involving the sum, difference or product of functions of a complex variable are the same as for functions of a real variable. In particular,
  - The limit of a product (sum) is the product (sum) of the limits.
  - The product and quotient rules for differentiation still apply.
  - The chain rule still applies.
- **Examples:**
  - Find $\dfrac{d}{dz}\left(\dfrac{z^2+1}{z-i}\right)$.
  - Find $\dfrac{d}{dz}\left(z^3+9z-7\right)^4$.

4. Analytic functions

- A function $f(z)$ is analytic at $z = z_0$ if $f(z)$ is differentiable in a neighborhood of z_0.
- A region of the complex plane is a set consisting of an open set, possibly together with some or all of the points on its boundary.
- We say that f is analytic in a region $\mathcal{R}$ of the complex plane, if it is analytic at every point in $\mathcal{R}$.
- One may use the word holomorphic instead of the word analytic.

Analytic functions (continued)

- A function that is analytic at every point in the complex plane is called entire.
- Polynomials of a complex variable are entire.
  - For instance, $f(z) = 3z - 7z^2 + z^3$ is analytic at every z.
- Rational functions of a complex variable of the form $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials, are analytic everywhere, except at the zeros of $h(z)$.
  - For instance, $\dfrac{z^2+1}{z-i}$ is analytic except at $z = i$.
  - In the above example, $z = i$ is called a pole of $f(z)$.

5. The Cauchy-Riemann equations

- If $f(z) = u(x, y) + iv(x, y)$ is defined in a neighborhood of $z = x + iy$, and if f is differentiable at z, then

$$u_x(x, y) = v_y(x, y), \qquad \text{and} \qquad u_y(x, y) = -v_x(x, y). \quad (1)$$

  These are called the Cauchy-Riemann equations.

- Conversely, if the partial derivatives of u and v exist in a neighborhood of $z = x + iy$, if they are continuous at z and satisfy the Cauchy-Riemann equations at z, then $f(z) = u(x, y) + iv(x, y)$ is differentiable at z.

- The Cauchy-Riemann equations therefore give a criterion for analyticity.

The Cauchy-Riemann equations (continued)

- Indeed, if a function is analytic at z, it must satisfy the Cauchy-Riemann equations in a neighborhood of z. In particular, if f does not satisfy the Cauchy-Riemann equations, then f cannot be analytic.
- Conversely, if the partial derivatives of u and v exist, are continuous, and satisfy the Cauchy-Riemann equations in a neighborhood of $z = x + iy$, then $f(z) = u(x, y) + iv(x, y)$ is analytic at z.
- **Examples:**
    - Use the Cauchy-Riemann equations to show that $\bar{z}$ is not analytic.
    - Use the Cauchy-Riemann equations to show that $\dfrac{1}{z}$ is analytic everywhere except at $z = 0$.

Applications of the Cauchy-Riemann equations

- A consequence of the Cauchy-Riemann equations is that

$$f'(z) = u_x + iv_x = v_y - iu_y. \tag{2}$$

- We will use these formulas later to calculate the derivative of some analytic functions.

- Another consequence of the Cauchy-Riemann equations is that an entire function with constant absolute value is constant. In fact, a more general result is that an entire function that is bounded (including at infinity) is constant.

6. Harmonic functions

- One can show that if f is analytic in a region $\mathcal{R}$ of the complex plane, then it is infinitely differentiable at any point in $\mathcal{R}$.
- If $f(z) = u(x, y) + iv(x, y)$ is analytic in $\mathcal{R}$, then both u and v satisfy Laplace's equation in $\mathcal{R}$, i.e.

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad \text{and} \qquad \nabla^2 v = v_{xx} + v_{yy} = 0. \quad (3)$$

- A function that satisfies Laplace's equation is called an harmonic function.

Harmonic conjugate

- If $f(z) = u(x, y) + iv(x, y)$ is analytic in $\mathcal{R}$, then we saw that both u and v are harmonic (i.e. satisfy Laplace's equation) in $\mathcal{R}$.
- We say that u and v are harmonic conjugates of one another.
- Given an harmonic function u, one can use the Cauchy-Riemann equations to find its harmonic conjugate v, and vice-versa.
- **Examples:**
  - Check that $u(x, y) = 2xy$ is harmonic, and find its harmonic conjugate v.
  - Given an harmonic function $v(x, y)$, how would you find its harmonic conjugate $u(x, y)$?