Chapter 13: Complex Numbers

Sections 13.5, 13.6 & 13.7

1. Complex exponential

- The exponential of a complex number $z = x + iy$ is defined as

$$\begin{aligned} \exp(z) &= \exp(x+iy) = \exp(x)\exp(iy) \\ &= \exp(x)\left(\cos(y) + i\sin(y)\right). \end{aligned}$$

- As for real numbers, the exponential function is equal to its derivative, i.e.

$$\frac{d}{dz}\exp(z) = \exp(z). \tag{1}$$

- The exponential is therefore entire.
- You may also use the notation $\exp(z) = e^z$.

Properties of the exponential function

- The exponential function is periodic with period $2\pi i$: indeed, for any integer $k \in \mathbb{Z}$,

$$\begin{aligned} \exp(z + 2k\pi i) &= \exp(x)\left(\cos(y + 2k\pi) + i\sin(y + 2k\pi)\right) \\ &= \exp(x)\left(\cos(y) + i\sin(y)\right) = \exp(z). \end{aligned}$$

- Moreover,

$$\begin{aligned} |\exp(z)| &= |\exp(x)|\,|\exp(iy)| = \exp(x)\sqrt{(\cos^2(y) + \sin^2(y))} \\ &= \exp(x) = \exp\left(\Re e(z)\right). \end{aligned}$$

- As with real numbers,
  - $\exp(z_1 + z_2) = \exp(z_1)\ \exp(z_2)$;
  - $\exp(z) \neq 0$.

2. Trigonometric functions

- The complex sine and cosine functions are defined in a way similar to their real counterparts,

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2}$$

- The tangent, cotangent, secant and cosecant are defined as usual. For instance,

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \qquad \sec(z) = \frac{1}{\cos(z)}, \qquad \text{etc.}$$

Trigonometric functions (continued)

- The rules of differentiation that you are familiar with still work.
- **Example:**
  - Use the definitions of $\cos(z)$ and $\sin(z)$,

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

    to find $(\cos(z))'$ and $(\sin(z))'$.
- Show that Euler's formula also works if θ is complex.

3. Hyperbolic functions

- The complex hyperbolic sine and cosine are defined in a way similar to their real counterparts,

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}. \tag{3}$$

- The hyperbolic sine and cosine, as well as the sine and cosine, are entire.

- We have the following relations

$$\begin{aligned} \cosh(iz) &= \cos(z), & \sinh(iz) &= i\,\sin(z), \\ \cos(iz) &= \cosh(z), & \sin(iz) &= i\,\sinh(z). \end{aligned} \tag{4}$$

4. Complex logarithm

- The logarithm w of $z \neq 0$ is defined as

$$e^w = z.$$

- Since the exponential is $2\pi i$-periodic, the complex logarithm is multi-valued.
- Solving the above equation for $w = w_r + iw_i$ and $z = re^{i\theta}$ gives

$$e^w = e^{w_r} e^{iw_i} = re^{i\theta} \Longrightarrow \begin{cases} e^{w_r} = r \\ w_i = \theta + 2p\pi \end{cases},$$

which implies $w_r = \ln(r)$ and $w_i = \theta + 2p\pi, \quad p \in \mathbb{Z}$.

- Therefore,

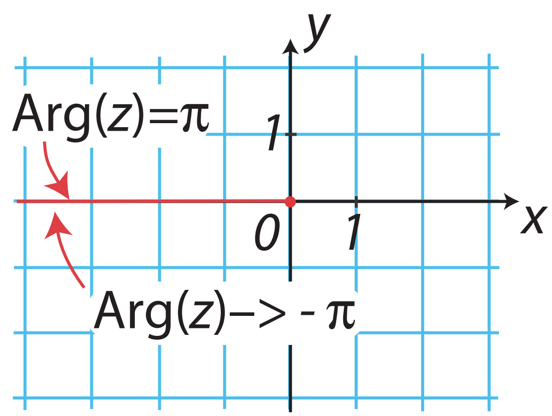
$$\ln(z) = \ln(|z|) + i\,\arg(z).$$

Principal value of $\ln(z)$

- We define the principal value of $\ln(z)$, $\text{Ln}(z)$, as the value of $\ln(z)$ obtained with the principal value of $\arg(z)$, i.e.

$$\text{Ln}(z) = \ln(|z|) + i\,\text{Arg}(z).$$

- Note that $\text{Ln}(z)$ jumps by $-2\pi i$ when one crosses the negative real axis from above.



- The negative real axis is called a branch cut of $\text{Ln}(z)$.

Principal value of $\ln(z)$ (continued)

- Recall that

$$\mathrm{Ln}(z) = \ln(|z|) + i\,\mathrm{Arg}(z).$$

- Since $\mathrm{Arg}(z) = \arg(z) + 2p\pi,\ p \in \mathbb{Z}$, we therefore see that $\ln(z)$ is related to $\mathrm{Ln}(z)$ by

$$\ln(z) = \mathrm{Ln}(z) + i\,2p\pi, \qquad p \in \mathbb{Z}.$$

- **Examples:**
  - $\mathrm{Ln}(2) = \ln(2)$, but $\ln(2) = \ln(2) + i\,2p\pi, \qquad p \in \mathbb{Z}$.
  - Find $\mathrm{Ln}(-4)$ and $\ln(-4)$.
  - Find $\ln(10\,i)$.

Properties of the logarithm

- You have to be careful when you use identities like

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2), \qquad \text{or} \qquad \ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2).$$

They are only true up to multiples of $2\pi i$.

- For instance, if $z_1 = i = \exp(i\pi/2)$ and $z_2 = -1 = \exp(i\pi)$,

$$\ln(z_1) = i\frac{\pi}{2} + 2p_1 i\pi, \qquad \ln(z_2) = i\pi + 2p_2 i\pi, \qquad p_1, p_2 \in \mathbb{Z},$$

and

$$\ln(z_1\, z_2) = i\frac{3\pi}{2} + 2p_3 i\pi, \qquad p_3 \in \mathbb{Z},$$

but p_3 is not necessarily equal to $p_1 + p_2$.

Properties of the logarithm (continued)

- Moreover, with $z_1 = i = \exp(i\pi/2)$ and $z_2 = -1 = \exp(i\pi)$,

$$\mathrm{Ln}(z_1) = i\,\frac{\pi}{2}, \quad \mathrm{Ln}(z_2) = i\,\pi,$$

and

$$\mathrm{Ln}(z_1\,z_2) = -i\,\frac{\pi}{2} \neq \mathrm{Ln}(z_1) + \mathrm{Ln}(z_2).$$

- However, every branch of the logarithm (i.e. each expression of $\ln(z)$ with a given value of $p \in \mathbb{Z}$) is analytic except at the branch point $z = 0$ and on the branch cut of $\ln(z)$. In the domain of analyticity of $\ln(z)$,

$$\frac{d}{dz}\left(\ln(z)\right) = \frac{1}{z}. \tag{5}$$

5. Complex power function

- If $z \neq 0$ and c are complex numbers, we define

$$\begin{aligned} z^c &= \exp\left(c \ \ln(z)\right) \\ &= \exp\left(c \operatorname{Ln}(z) + 2pc\pi i\right), \qquad p \in \mathbb{Z}. \end{aligned}$$

- For $c \in \mathbb{C}$, this is again a multi-valued function, and we define the principal value of z^c as

$$z^c = \exp\left(c \operatorname{Ln}(z)\right)$$