Chapter 5: Expansions

Sections 5.1, 5.2, 5.7 & 5.8

1. Power series solutions of ordinary differential equations

- A power series about $x = x_0$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- This series is convergent (or converges) if the sequence of partial sums

$$S_n(x) = \sum_{i=0}^{n} a_i (x - x_0)^i$$

has a (finite) limit, $S(x)$, as $n \to \infty$. In such a case, we write

$$S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- If the series is not convergent, we say that it is divergent, or that it diverges.

Radius of convergence

- One can show (Abel's lemma) that if a power series converges for $|x - x_0| = R_0$, then it converges absolutely for all x's such that $|x - x_0| < R_0$.
- This allows us to define the radius of convergence R of the series as follows:
  - If the series only converges for $x = x_0$, then $R = 0$.
  - If the series converges for all values of x, then $R = \infty$.
  - Otherwise, R is the largest number such that the series converges for all x's that satisfy $|x - x_0| < R$.
- A useful test for convergence is the ratio test:

$$R = \frac{1}{K}, \text{ where } K = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

where K could be infinite or zero, and it is assumed that the a_n's are non-zero.

Power series as solutions to ODE's

- Taylor series are power series.
- A function f is analytic at a point $x = x_0$ if it can locally be written as a convergent power series, i.e. if there exists $R > 0$ such that
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
for all x's that satisfy $|x - x_0| < R$.
- If the functions p/h and q/h in the differential equation
$$h(x)y'' + p(x)y' + q(x) = 0 \qquad (1)$$
are analytic at $x = x_0$, then every solution of (1) is analytic at $x = x_0$.

Power series as solutions to ODE's (continued)

- We can therefore look for solutions to (1) in the form of a power series.
- **Example:** Solve $y'' - 2y' + y = 0$ by the power series method.
- Many special functions are defined as power series solutions to differential equations like (1).
  - Legendre polynomials are solutions to Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ where n is a non-negative integer.
  - Bessel functions are solutions to Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ with $\nu \in \mathbb{C}$.

Separation of variables for the wave equation

- Consider the wave equation on a string of length L:

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial x}$$

for a function $u(x,t)$ defined on a rectangle $[0,L] \times [0,T_f]$.

- We may try to solve the equation first by assuming $u(x,t) = X(x)\,T(t)$, where X and T are functions of one variable. Plugging this into the equation we get

$$X(x)\,T''(t) = X''(x)\,T(t)\,,$$

or

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Wave equation, cont'd

- Since one side is a function of t alone and the other is a function of x alone, we find that both must be equal to some constant, i.e.,

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$$

  for some constant k. (We don't know what k is!)
- If we fix the endpoints of the string, say $u(0,t) = u(L,t) = 0$, then in order to solve the wave equation, we must solve the following boundary value problem:

$$X'' - kX = 0, \quad X(0) = X(L) = 0.$$

  for different values of k.
- This is a Sturm-Liouville problem.

2. Sturm-Liouville problems

- A regular Sturm-Liouville problem is an eigenvalue problem of the form

$$L\, y = -\lambda\, \sigma(x)\, y, \qquad L\, y = \left[p(x)y'\right]' + q(x)y, \qquad (2)$$

  p, q and σ are real continuous functions on $[a, b]$, a, $b \in \mathbb{R}$, $p(x) > 0$ and $\sigma(x) > 0$ on $[a, b]$, and $y(x)$ is square-integrable on $[a, b]$ and satisfies given boundary conditions.

- In what follows, we will use separated boundary conditions

$$C_1 y(a) + C_2 y'(a) = 0, \qquad C_3 y(b) + C_4 y'(b) = 0. \qquad (3)$$

- An eigenvalue of the Sturm-Liouville problem is a number λ for which there exists an eigenfunction $y(x) \neq 0$ that satisfies (2) and (3).

Sturm-Liouville problems (continued)

- One can show that with separated boundary conditions, all eigenvalues of the Sturm-Liouville problem are real (assuming they exist).
- In such a case, eigenfunctions associated with different eigenvalues are orthogonal (with respect to the weight function σ).
- Two functions $y_1(x)$ and $y_2(x)$ are orthogonal with respect to the weight function σ ($\sigma(x) > 0$ on $[a, b]$) if

$$< y_1, y_2 > \equiv \int_a^b y_1(x)\, y_2(x)\, \sigma(x)\; dx = 0.$$

Sturm-Liouville problems (continued)

- Legendre's and Bessel's equations are examples of singular Sturm-Liouville problems.
- Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ can be written as
$$\left[p(x)y'\right]' + q(x)y = -\lambda y$$
where $p(x) = 1 - x^2$, $q(x) = 0$ and $\lambda = n(n+1)$. In this case there are no boundary conditions and $[a, b] = [-1, 1]$.
- Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ can be written in the form (2) by setting $p(x) = \sigma(x) = x$, $\lambda = 1$, and $q(x) = -\nu^2/x$. In this case, $[a, b] = [0, R]$, $R > 0$ and $y(x)$ is required to vanish at $x = R$.

3. Orthogonal eigenfunction expansions

- Recall that if A is a square $n \times n$ matrix with real entries, then the (genuine and generalized) eigenvectors of A, $U_1, U_2, \cdots, U_n$, form a basis of $\mathbb{R}^n$.
- This means that every vector $X \in \mathbb{R}^n$ can be written in the form
$$X = a_1 U_1 + a_2 U_2 + \cdots + a_n U_n, \tag{4}$$
where the coefficients a_i are uniquely determined.
- Moreover, if the U_i's are orthonormal (i.e. orthogonal and of norm one), then each coefficient a_i can be found by taking the dot product of X with U_i, i.e. $a_i =< X, U_i >$.
- In this case, (4) is an orthogonal expansion of X on the eigenvectors of A.

Orthogonal eigenfunction expansions (continued)

- Similarly, there exist special linear differential operators, such as Sturm-Liouville operators, whose eigenfunctions form a complete orthonormal basis for a space of functions satisfying given boundary conditions.
- We can then use such a complete orthonormal basis, $\{y_1, y_2, \cdots\}$, to write any function in the space as a uniquely determined linear combination of the basis functions. Such an expansion is called an orthonormal expansion or a generalized Fourier series.
- In such a case, for every function f in the space, we can write

$$f(x) = \sum_{i=1}^{\infty} a_i\, y_i(x), \qquad a_i = < f, y_i >, \qquad ||y_i|| = 1.$$

Trigonometric series

- Trigonometric series are the most important example of Fourier series.
- Consider the Sturm-Liouville problem with periodic boundary conditions $(p(x) = 1,\ q(x) = 0,\ \sigma(x) = 1)$,

$$y'' + \lambda y = 0, \qquad y(\pi) = y(-\pi), \qquad y'(\pi) = y'(-\pi).$$

- The eigenfunctions are 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$, $\cdots$, $\cos(mx)$, $\sin(mx)$, $\cdots$, and correspond to the eigenvalues $0, 1, 1, 4, 4, \cdots, m^2, m^2, \cdots$.
- The above eigenfunctions are orthogonal but not of norm one. They can be made orthonormal by dividing each eigenfunction by its norm. They form a complete basis of the space of square integrable functions on $[-\pi, \pi]$.