Chapter 11: Fourier Series

Sections 1 - 5

1. Fourier series

- We saw before that the set of functions $\{1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \cdots,\ \cos(mx),\ \sin(mx),\ \cdots\}$, where m is a non-negative integer, forms a complete orthogonal basis of the space of square integrable functions on $[-\pi, \pi]$.
- This means that we can define the Fourier series of any square integrable function on $[-\pi, \pi]$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

where $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\ dx$ and, for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx.$$

Convergence of Fourier series

- If f is continuously differentiable on $[-\pi, \pi]$ except at possibly a finite number of points where it has a left-hand and a right-hand derivative, then the partial sum

$$f_N(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$

with the a_i defined above, converges to $f(x)$ as $N \to \infty$ if f is continuous at x. At a point of discontinuity, the Fourier series converges towards

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right].$$

Convergence of Fourier series (continued)

- **Examples:**
  - Calculate the first three non-zero Fourier coefficients of the rectangular wave function

$$f(x) = \begin{cases} -\frac{\pi}{4} & \text{if } -\pi < x \leq 0 \\ \frac{\pi}{4} & \text{if } 0 < x \leq \pi \end{cases} \qquad \text{and} \qquad f(x+2\pi) = f(x).$$

  - To what value does the above Fourier series converge if
    - $x = 0$?
    - $x = 1$?
    - $x = \pi$?
  - Experiment with the MIT applet called *Fourier Coefficients*.
- Gibbs phenomenon: Near a point of discontinuity x_0, the partial sums $f_N(x)$ exhibits oscillations which, for small values of N, are noticeable even far from x_0. As $N \to \infty$, the oscillations get "compressed" near x_0 but never disappear.

2. Fourier series for $2L$-periodic functions

- If instead of being 2π-periodic, the function f has period $2L$, we can obtain its Fourier series by re-scaling the variable x.
- Indeed, let $g(v) = f\left(\frac{vL}{\pi}\right)$. Then, g is 2π-periodic and one can write down its Fourier series as before. Going back to the x-variable, one obtains

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{\pi x}{L}\right) + b_n \sin\left(n\frac{\pi x}{L}\right)\right],$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$ and, for $n \geq 1$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi x}{L}\right)\, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi x}{L}\right)\, dx.$$

3. Even and odd functions

From the above formula, it is easy to see that

- If f is even, then the b_n's are all zero, and the Fourier series of f is a Fourier cosine series, i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{\pi x}{L}\right)\right].$$

  Its non-zero coefficients are given by

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi x}{L}\right)\, dx.$$

- Similarly, if f is odd, then the a_n's are all zero, and the Fourier series of f is a Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty}\left[b_n \sin\left(n\frac{\pi x}{L}\right)\right], \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi x}{L}\right)\, dx.$$

4. Complex form of the Fourier series

- The Fourier series of a function f,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\frac{\pi x}{L} \right) + b_n \sin\left(n\frac{\pi x}{L} \right) \right],$$

  can be re-written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i\, n\frac{\pi x}{L} \right),$$

  where the complex coefficients c_n are given by

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left(-i\, n\frac{\pi x}{L} \right)\, dx, \qquad n = 0,\ \pm 1,\ \pm 2,\ \cdots .$$

5. Half-range expansions

- Sometimes, if one only needs a Fourier series for a function defined on the interval $[0, L]$, it may be preferable to use a sine or cosine Fourier series instead of a regular Fourier series.
- This can be accomplished by extending the definition of the function in question to the interval $[-L, 0]$ so that the extended function is either even (if one wants a cosine series) or odd (if one wants a sine series).
- Such Fourier series are called half-range expansions.
- **Example:** Find the half-range sine and cosine expansions of the function $f(x) = 1$ on the interval $[0, 1]$.

6. Forced oscillations

- Consider the forced and damped oscillator described by $ay'' + by' + cy = f(x)$, where $b^2 - 4ac < 0$, b is positive and small, and f is a periodic forcing function.
- We know that the general solution to this equation is the sum of a particular solution and the general solution to the homogeneous equation, i.e. $y(x) = y_h(x) + y_p(x)$.
- Since the equation is linear, the principle of superposition applies. Using Fourier series, we can think of f as a superposition of sines and cosines. As a consequence, if one of the terms in the forcing has a frequency close to the natural frequency of the oscillator, one can expect the solution to be dominated by the corresponding mode.
- See the MIT applet called *Harmonic Frequency Response*.