Chapter 11: Fourier Transforms

Sections 8 & 9

1. Fourier transforms

- Consider a function f, which is not necessarily periodic, but absolutely integrable (i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$) and piecewise continuously differentiable on $(-\infty, \infty)$.
- The Fourier transform of f is defined as

$$\mathcal{F}(f) = \widehat{f}, \qquad \text{where} \qquad \widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx.$$

- The inverse Fourier transform of $\widehat{f}$ is defined as

$$\mathcal{F}^{-1}(\widehat{f}) = f, \qquad \text{where} \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) \exp(ikx)\, dk.$$

- The relation $f = \mathcal{F}^{-1}(\mathcal{F}(f))$ reads

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\nu) \exp(ik(x-\nu))\, d\nu\, dk. \qquad (1)$$

Properties of the Fourier transform

- As for Fourier series, Equation (1), i.e. $f(x) = \left(\mathcal{F}^{-1}(\widehat{f}\,)\right)(x)$ is only true at points where f is continuous.
- At a point of discontinuity x_0 of f, the inverse Fourier transform of f converges to the average $\frac{1}{2}\left[f^{+}(x_0) + f^{-}(x_0)\right]$.
- The Fourier transform is a linear transformation, i.e. if f_1 and f_2 are such that their Fourier transforms exist and if α and β are two arbitrary constants, then

$$\mathcal{F}(\alpha\, f_1 + \beta\, f_2) = \alpha\, \mathcal{F}(f_1) + \beta\, \mathcal{F}(f_2)$$

- Fourier transform of the derivative. If f and its derivatives are piecewise continuously differentiable and are absolutely integrable on $\mathbb{R}$, and if $lim_{x\to\pm\infty} f(x) = 0$, then the Fourier transform of the derivative of f is such that $\widehat{f'}(k) = ik\,\widehat{f}\,(k)$.

Convolution

- The convolution of two absolutely integrable functions f and g is denoted by $f * g$ and defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\ dt = \int_{-\infty}^{\infty} f(t)g(x-t)\ dt.$$

- Convolution theorem. If f and g are both piecewise continuously differentiable and absolutely integrable on $\mathbb{R}$, then the Fourier transform of the convolution of f and g is given by

$$\mathcal{F}(f * g) = \sqrt{2\pi}\ \mathcal{F}(f)\ \mathcal{F}(g).$$

- **Example:** Find the Fourier transform of $f * g$ where $f(x) = \exp(-ax^2)$, $a > 0$, and g is such that $g(x) = \exp(-ax)$ if $x > 0$ and $g(x) = 0$ otherwise.

2. Sine and cosine transforms

Consider a piecewise continuously differentiable function f, which is absolutely integrable on $\mathbb{R}$.

- If f is even, then the Fourier transform of f can be written as a cosine transform, i.e.

$$\widehat{f}(k) = \widehat{f}_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(kx) \, dx,$$

  and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_c(k) \cos(kx) \, dk.$$

- Similarly, if f is odd, then the Fourier transform of f is a sine transform, i.e. $\widehat{f}(k) = -i\,\widehat{f}_s(k)$, where

$$\widehat{f}_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(kx) \, dx, \; f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_s(k) \sin(kx) \, dk.$$