# Chapters 7-8: Linear Algebra

## Sections 7.1, 7.2 & 7.4





## 1. Matrices and vectors

- An $m \times n$ matrix is an array with $m$ rows and $n$ columns. It is typically written in the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $i$ is the row index and $j$ is the column index.

- A column vector is an $m \times 1$ matrix. Similarly, a row vector is a $1 \times n$ matrix.
- The entries $a_{ij}$ of a matrix $A$ may be real or complex.

## Matrices and vectors (continued)

- **Examples:**
  - $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a $2 \times 2$ square matrix with real entries.
  - $u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a column vector of $A$.
  - $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 3-7i \end{bmatrix}$ is a $3 \times 3$ diagonal matrix, with complex entries.
  - An $n \times n$ diagonal matrix whose entries are all ones is called the $n \times n$ identity matrix.
  - $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$ is a $2 \times 4$ matrix with real entries.





## Matrix addition and scalar multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices, and let $c$ be a scalar.

- The matrices $A$ and $B$ are equal if and only if they have the same entries,

$$A = B \iff a_{ij} = b_{ij}, \text{ for all } i, j, \ 1 \le i \le m, \ 1 \le j \le n.$$

- The sum of $A$ and $B$ is the $m \times n$ matrix obtained by adding the entries of $A$ to those of $B$,

$$A + B = [a_{ij} + b_{ij}] \,.$$

- The product of $A$ with the scalar $c$ is the $m \times n$ matrix obtained by multiplying the entries of $A$ by $c$,

$$c\,A = [c\,a_{ij}] \,.$$

## 2. Matrix multiplication

- Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. The product $C = AB$ of $A$ and $B$ is an $m \times p$ matrix whose entries are obtained by multiplying each row of $A$ with each column of $B$ as follows:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}.$$

- **Examples:** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$.
  - Is the product $AC$ defined? If so, evaluate it.
  - Same question with the product $CA$.
  - What is the product of $A$ with the third column vector of $C$?





## Matrix multiplication (continued)

- **More examples:**
  - Consider the system of equations

$$\begin{cases} 3x_1 + 2x_2 - x_3 = 4 \\ x_2 - 7x_3 = 0 \\ -x_1 + 4x_2 - 6x_3 = -10 \end{cases}.$$

    Write this system in the form $AX = Y$, where $A$ is a matrix and $X$ and $Y$ are two column vectors.
  - Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

    Calculate the products $AB$ and $BA$.

## 3. Rules for matrix addition and multiplication

- The rules for matrix addition and multiplication by a scalar are the same as the rules for addition and multiplication of real or complex numbers.
- In particular, if $A$ and $B$ are matrices and $c_1$ and $c_2$ are scalars, then

$$
\begin{aligned}
&A+B=B+A\\
&(A+B)+C=A+(B+C)\\
&c_1\,(A+B)=c_1\,A+c_1\,B\\
&(c_1+c_2)A=c_1A+c_2A\\
&c_1\,(c_2\,A)=(c_1\,c_2)A
\end{aligned}
$$

whenever the above quantities make sense.





## Rules for matrix addition and multiplication (continued)

- The product of two matrices is associative and distributive, i.e.

$$
\begin{aligned}
&A(BC)=(AB)C=ABC\\
&A(B+C)=AB+AC \qquad (A+B)C=AC+BC.
\end{aligned}
$$

- However, the product of two matrices is not commutative. If $A$ and $B$ are two square matrices, we typically have

$$AB\neq BA$$

- For two square matrices $A$ and $B$, the commutator of $A$ and $B$ is defined as

$$[A,B]=AB-BA.$$

In general, $[A,B]\neq 0$. If $[A,B]=0$, one says that the matrices $A$ and $B$ commute.

# 4. Transposition

- The transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ obtained from $A$ by switching its rows and columns, i.e.

$$\text{if} \quad A = [a_{ij}], \qquad \text{then} \quad A^T = [a_{ji}].$$

- **Example:** Find the transpose of $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$.
- Some properties of transposition. If $A$ and $B$ are matrices, and $c$ is a scalar, then

$$(A+B)^T = A^T + B^T \qquad (c\,A)^T = c\,A^T$$
$$(A\,B)^T = B^T A^T \qquad \left(A^T\right)^T = A,$$

whenever the above quantities make sense.





# 5. Linear independence

- A linear combination of the $n$ vectors $a_1, \ a_2, \ \cdots, \ a_n$ is an expression of the form

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n,$$

where the $c_i$'s are scalars.

- A set of vectors $\{a_1, a_2, \cdots, a_n\}$ is linearly independent if the only way of having a linear combination of these vectors equal to zero is by choosing all of the coefficients equal to zero. In other words, $\{a_1, a_2, \cdots, a_n\}$ is linearly independent if and only if

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0 \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

## Linear independence (continued)

- **Examples:**
  - Are the columns of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ linearly independent?
  - Same question with the columns of the matrix $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$.
  - Same question with the rows of the matrix $C$ defined above.
- A set that is not linearly independent is called linearly dependent.
- Can you find a condition on a set of $n$ vectors, which would guarantee that these vectors are linearly dependent?





## 6. Vector space

- A real (or complex) vector space is a non-empty set $V$ whose elements are called vectors, and which is equipped with two operations called vector addition and multiplication by a scalar.
- The vector addition satisfies the following properties.
  1. The sum of two vectors $a \in V$ and $b \in V$ is denoted by $a + b$ and is an element of $V$.
  2. It is commutative: $a + b = b + a$, for all $a, b \in V$.
  3. It is associative: $(a + b) + c = a + (b + c)$ for all $a, b, c \in V$.
  4. There exists a unique zero vector, denoted by 0, such that for every vector $a \in V$, $a + 0 = a$.
  5. For each $a \in V$, there exists a unique vector $(-a) \in V$ such that $a + (-a) = 0$.

## Vector space (continued)

- The multiplication by a scalar satisfies the following properties.
  1. The multiplication of a vector $a \in V$ by a scalar $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$) is denoted by $\alpha\, a$ and is an element of $V$.
  2. Multiplication by a scalar is distributive:
  $$\alpha\,(a+b) = \alpha\, a + \alpha\, b, \qquad (\alpha+\beta)\, a = \alpha\, a + \beta\, a,$$
  for all $a, b \in V$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$).
  3. It is associative: $\alpha\,(\beta a) = (\alpha\,\beta)\, a$ for all $a \in V$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$).
  4. Multiplying a vector by 1 gives back that vector, i.e.
  $$1\, a = a,$$
  for all $a \in V$.





## Bases and dimension

- The span of set of vectors $\mathcal{U} = \{a_1, a_2, \cdots, a_n\}$ is the set of all linear combinations of vectors in $\mathcal{U}$. It is denoted by
$$\text{Span}\{a_1, a_2, \cdots, a_n\} \text{ or } \text{Span}(\mathcal{U})$$
and is a subspace of $V$.
- A basis $\mathcal{B}$ of a subspace $S$ of $V$ is a set of vectors of $S$ such that
  1. $\text{Span}(\mathcal{B}) = S$;
  2. $\mathcal{B}$ is a linearly independent set.
- Theorem: If a basis $\mathcal{B}$ of a subspace $S$ of $V$ has $n$ vectors, then all other bases of $S$ have exactly $n$ vectors.
- The dimension of a vector space $V$ (or of a subspace $S$ of $V$) spanned by a finite number of vectors is the number of vectors in any of its bases.

# 7. Rank

- The row space of an $m \times n$ matrix $A$ is the span of the row vectors of $A$. If $A$ has real entries, the row space of $A$ is a subspace of $\mathbb{R}^n$.
- Similarly, the column space of $A$ is the span of the column vectors of $A$, and is a subspace of $\mathbb{R}^m$.
- The rank of a matrix $A$ is the dimension of its column space.
- Theorem: The dimensions of the row and column spaces of a matrix $A$ are the same. They are equal to the rank of $A$.
- **Example:** Check that the row and column spaces of $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$ are vector subspaces, and find their dimension.



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Definitions
The rank theorem

# The rank theorem

- The null space of an $m \times n$ matrix $A$, $\mathcal{N}(A)$ is the set of vectors $u$ such that $Au = 0$. If $A$ has real entries, then $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.
- The rank theorem states that if $A$ is an $m \times n$ matrix, then

$$\text{rank}(A) + \dim\left(\mathcal{N}(A)\right) = n.$$

- **Example:** Find the rank and the null space of the matrix $C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 1 & 6 & -8 & 0 \end{bmatrix}$. Check that the rank theorem applies.

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