# Chapters 7-8: Linear Algebra

## Sections 7.5, 7.8 & 8.1

# 1. Linear systems of equations

- A linear system of equations of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
\cdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

can be written in matrix form as $AX = B$, where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m \end{bmatrix}
$$

## Solution(s) of a linear system of equations

1. Given a matrix $A$ and a vector $B$, a solution of the system $AX = B$ is a vector $X$ which satisfies the equation $AX = B$.
2. If $B$ is not in the column space of $A$, then the system $AX = B$ has no solution. One says that the system is not consistent. In the statements below, we assume that the system $AX = B$ is consistent.
3. If the null space of $A$ is non-trivial, then the system $AX = B$ has more than one solution.
4. The system $AX = B$ has a unique solution provided $\dim(\mathcal{N}(A)) = 0$.
5. Since, by the rank theorem, $\text{rank}(A) + \dim(\mathcal{N}(A)) = n$ (recall that $n$ is the number of columns of $A$), the system $AX = B$ has a unique solution if and only if $\text{rank}(A) = n$.

# Row operations.

- There are three types of row operations:
  1. Multiply a nonzero constant times an entire row. $(r_i \longrightarrow ar_i)$
  2. Exchange rows. $(r_i \longrightarrow r_j$ and $r_j \longrightarrow r_i)$
  3. Add a multiple of one row to another. $(r_i \longrightarrow ar_j + r_i)$
- Row operations do not change the span of the row space.
- There are corresponding column operations, which do not change the column space.

# Row operations to solve linear systems.

Row operations can be used to solve a linear system $AX = B$

$$\begin{aligned} -x - 4y + z &= 10 \\ x + y - 2z &= 2 \\ 2x - y - 5z &= 3 \end{aligned}$$

- Write an augmented matrix $(A|B)$.

$$\left(\begin{array}{ccc|c} -1 & -4 & 1 & 10 \\ 1 & 1 & -2 & 2 \\ 2 & -1 & -5 & 16 \end{array}\right)$$

- Use row operations to get zeroes in the first column:

$$\left(\begin{array}{ccc|c} -1 & -4 & 1 & 10 \\ 0 & -3 & -1 & 12 \\ 0 & -9 & -3 & 36 \end{array}\right) \begin{array}{c} \\ r_1 + r_2 \\ 2r_1 + r_3 \end{array}$$

$$\left(\begin{array}{ccc|c} -1 & -4 & 1 & 10 \\ 0 & -3 & -1 & 12 \\ 0 & -9 & -3 & 36 \end{array}\right)$$

- Do the same with the next column:

$$\left(\begin{array}{ccc|c} -1 & -4 & 1 & 10 \\ 0 & -3 & -1 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ -3r_2 + r_3 \end{array}$$

- This is equivalent to the simplified system

$$\begin{aligned} -x - 4y + z &= 10 \\ -3y - z &= 12 \\ 0 &= 0 \end{aligned}$$

- To solve the system, use back substitution.

# Row operations to compute the rank of a matrix.

- Given a matrix $A$, row operations do not change the row space.
- Since the matrix

$$A = \begin{pmatrix} -1 & -4 & 1 & 10 \\ 1 & 1 & -2 & 2 \\ 2 & -1 & -5 & 16 \end{pmatrix}$$

can be made into the matrix

$$A' = \begin{pmatrix} -1 & -4 & 1 & 10 \\ 0 & -3 & -1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

by doing row operations, the two matrices have the same row spaces.
- It is easy to see that the first two rows are linearly independent, so the rank is 2.

# Consistency

- The system $AX = B$ is consistent, i.e., has a solution if (equivalently):
  1. Gaussian elimination on the augmented matrix $(A|B)$ yields a matrix of the form:

$$\left(\begin{array}{cccccccc|c} a_1 & * & * & & & & & & b_1 \\ 0 & a_2 & * & * & & & & & b_2 \\ 0 & 0 & 0 & a_3 & * & * & & & \dots \\ 0 & 0 & 0 & 0 & \dots & * & * & & \\ 0 & 0 & 0 & 0 & 0 & 0 & a_r & * & b_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

     i.e., any rows reduced to all zeroes before the line are also zero after the line.
  2. The rank of $(A|B)$ is equal to the rank of $A$.

# Inconsistency

- The system $AX = B$ is inconsistent, i.e., has NO SOLUTION if (equivalently):
  1. Gaussian elimination on the augmented matrix $(A|B)$yields a matrix of the form:

$$\left(\begin{array}{cccccccc|c} a_1 & * & & & & & & & b_1 \\ 0 & a_2 & * & * & & & & & b_2 \\ 0 & 0 & 0 & a_3 & * & & & & \dots \\ 0 & 0 & 0 & 0 & \dots & * & * & * & \\ 0 & 0 & 0 & 0 & 0 & 0 & a_r & * & b_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{r+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

     where $b_{r+1} \neq 0$, i.e., there is a row of zeroes before the line with a nonzero element after the line.
  2. The rank of $(A|B)$ is greater than the rank of $A$.
  3. The vector $B$ is not in the column space of $A$.

# Unique solutions

- The system $AX = B$ has one unique solution if (equivalently):
  1. Gaussian elimination on the augmented matrix $(A|B)$ yields a matrix of the form:
  $$\left(\begin{array}{ccccc|c} a_1 & & & & & b_1 \\ 0 & a_2 & & & & b_2 \\ 0 & 0 & a_3 & & & \dots \\ 0 & 0 & 0 & \dots & & \\ 0 & 0 & 0 & 0 & a_n & b_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$
  i.e., there are all nonzero numbers on the "diagonal."
  2. The rank of $A$ is equal $n$ (which is equal to the rank of $(A|B)$), which is the maximum rank (so it is essential that $n \geq m$). This means that $\dim(\mathcal{N}(A)) = 0$, i.e., the nullspace is trivial.
  3. The columns of $A$ form a basis for the column space.

# Infinitely many solutions

- The system $AX = B$ has lots of solutions if (equivalently):
  1. Gaussian elimination on the augmented matrix $(A|B)$ yields a matrix of the form:

$$\left(\begin{array}{cccccccc|c} a_1 & * & * & & & & & & b_1 \\ 0 & a_2 & * & * & & & & & b_2 \\ 0 & 0 & 0 & a_3 & * & * & & & \dots \\ 0 & 0 & 0 & 0 & \dots & * & * & & \\ 0 & 0 & 0 & 0 & 0 & a_r & * & * & b_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

     i.e., there are zeroes on the diagonal and/or the last diagonal nonzero element is not next to the line $|$.
  2. The rank of $A$ is less than $n$. This is equivalent to $\dim(\mathcal{N}(A)) > 0$.
  3. The columns of $A$ do not form a basis of the column space.

# Solution(s) of a linear system of equations (continued)

1. A linear system of the form $AX = 0$ is said to be homogeneous.
2. Solutions of $AX = 0$ are vectors in the null space of $A$.
3. If we know one solution $X_0$ to $AX = B$, then all solutions to $AX = B$ are of the form

$$X = X_0 + X_h$$

   where $X_h$ is a solution to the associated homogeneous equation $AX = 0$.
4. In other words, the general solution to the linear system $AX = B$, if it exists, can be written as the sum of a particular solution $X_0$ to this system, plus the general solution of the associated homogeneous system.

# 2. Inverse of a matrix

- If $A$ is a square $n \times n$ matrix, its inverse, if it exists, is the matrix, denoted by $A^{-1}$, such that

$$A\,A^{-1} = A^{-1}\,A = I_n,$$

where $I_n$ is the $n \times n$ identity matrix.

- A square matrix $A$ is said to be singular if its inverse does not exist. Similarly, we say that $A$ is non-singular or invertible if $A$ has an inverse.

- The inverse of a square matrix $A = [a_{ij}]$ is given by

$$A^{-1} = \frac{1}{\det(A)} [C_{ij}]^T,$$

where $\det(A)$ is the determinant of $A$ and $C_{ij}$ is the matrix of cofactors of $A$.

# Determinant of a matrix

- The determinant of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where the cofactor $C_{ij}$ is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

and the minor $M_{ij}$ is the determinant of the matrix obtained from $A$ by "deleting" the $i$-th row and $j$-th column of $A$.

- **Example:** Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

# Properties of determinants

- If a determinant has a row or a column entirely made of zeros, then the determinant is equal to zero.
- The value of a determinant does not change if one replaces one row (resp. column) by itself plus a linear combination of other rows (resp. columns).
- If one interchanges 2 columns in a determinant, then the value of the determinant is multiplied by $-1$.
- If one multiplies a row (or a column) by a constant $C$, then the determinant is multiplied by $C$.
- If $A$ is a square matrix, then $A$ and $A^T$ have the same determinant.

# Properties of the inverse

- Since the inverse of a square matrix $A$ is given by

$$A^{-1} = \frac{1}{\det(A)} \left[C_{ij}\right]^T ,$$

  we see that $A$ is invertible if and only if $\det(A) \neq 0$.

- If $A$ is an invertible $2 \times 2$ matrix, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

  and $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.

- If $A$ and $B$ are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{and} \qquad \left(A^{-1}\right)^{-1} = A.$$

# Linear systems of $n$ equations with $n$ unknowns

- Consider the following linear system of $n$ equations with $n$ unknowns,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\ldots & \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

- This system can be also be written in matrix form as $AX = B$, where $A$ is a square matrix.

- If $\det(A) \neq 0$, then the above system has a unique solution $X$ given by

$$X = A^{-1}B.$$

# Linear systems of equations - summary

Consider the linear system $AX = B$ where $A$ is an $m \times n$ matrix.

- The system may not be consistent, in which case it has no solution.
- To decide whether the system is consistent, check that $B$ is in the column space of $A$.
- If the system is consistent, then
  - Either $\text{rank}(A) = n$ (which also means that $\dim(\mathcal{N}(A)) = 0$), and the system has a unique solution.
  - Or $\text{rank}(A) < n$ (which also means that $\mathcal{N}(A)$ is non-trivial), and the system has an infinite number of solutions.

# Linear systems of equations - summary (continued)

Consider the linear system $AX = B$ where $A$ is an $m \times n$ matrix.

- If $m = n$ and the system is consistent, then
  - Either $\det(A) \neq 0$, in which case $\text{rank}(A) = n$, $\dim(\mathcal{N}(A)) = 0$, and the system has a unique solution;
  - Or $\det(A) = 0$, in which case $\dim(\mathcal{N}(A)) > 0$, $\text{rank}(A) < n$, and the system has an infinite number of solutions.
- Note that when $m = n$, having $\det(A) = 0$ means that the columns of $A$ are linearly dependent.
- It also means that $\mathcal{N}(A)$ is non-trivial and that $\text{rank}(A) < n$.

# 3. Eigenvalues and eigenvectors

- Let $A$ be a square $n \times n$ matrix. We say that $X$ is an eigenvector of $A$ with eigenvalue $\lambda$ if

$$X \neq 0 \qquad \text{and} \qquad AX = \lambda X.$$

- The above equation can be re-written as

$$(A - \lambda I_n)X = 0.$$

- Since $X \neq 0$, this implies that $A - \lambda I_n$ is not invertible, i.e. that $\det(A - \lambda I_n) = 0$.

- The eigenvalues of $A$ are therefore found by solving the characteristic equation $\det(A - \lambda I_n) = 0$.

# Eigenvalues

- The characteristic polynomial $\det(A - \lambda I_n)$ is a polynomial of degree $n$ in $\lambda$. It has $n$ complex roots, which are not necessarily distinct from one another.
- If $\lambda$ is a root of order $k$ of the characteristic polynomial $\det(A - \lambda I_n)$, we say that $\lambda$ is an eigenvalue of $A$ of algebraic multiplicity $k$.
- If $A$ has real entries, then its characteristic polynomial has real coefficients. As a consequence, if $\lambda$ is an eigenvalue of $A$, so is $\bar{\lambda}$.
- It $A$ is a $2 \times 2$ matrix, then its characteristic polynomial is of the form $\lambda^2 - \lambda \operatorname{Tr}(A) + \det(A)$, where the trace of $A$, $\operatorname{Tr}(A)$, is the sum of the diagonal entries of $A$.

# Eigenvalues (continued)

- **Examples:** Find the eigenvalues of the following matrices.
  - $A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.
  - $B = \begin{bmatrix} -1 & 9 \\ 0 & 5 \end{bmatrix}$.
  - $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$.
  - $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

# Eigenvectors

- Once an eigenvalue $\lambda$ of $A$ has been found, one can find an associated eigenvector, by solving the linear system

$$(A - \lambda I_n)\, X = 0.$$

- Since $\mathcal{N}(A - \lambda I_n)$ is not trivial, there is an infinite number of solutions to the above equation. In particular, if $X$ is an eigenvector of $A$ with eigenvalue $\lambda$, so is $\alpha X$, where $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $\alpha \neq 0$.

- The set of eigenvectors of $A$ with eigenvalue $\lambda$, together with the zero vector, form a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), $E_\lambda$, called the eigenspace of $A$ corresponding to the eigenvalue $\lambda$.

- The dimension of $E_\lambda$ is called the geometric multiplicity of $\lambda$.

# Eigenvectors (continued)

- **Examples:** Find the eigenvectors of the following matrices. Each time, give the algebraic and geometric multiplicities of the corresponding eigenvalues.
  - $A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.
  - $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$.
  - $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.