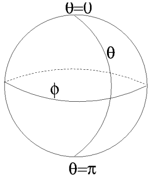
Eigenfunctions on the surface of a sphere

In spherical coordinates, the Laplacian is

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left[\frac{u_{\phi\phi}}{\sin^2(\theta)} + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta\right].$$



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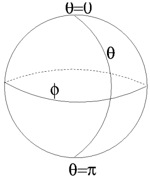

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Separating out the r variable, left with the eigenvalue problem for $v(\phi, \theta)$

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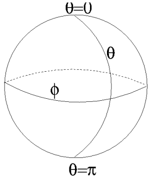

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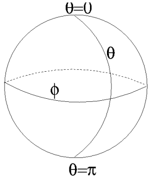

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The problem for q is familiar: $q''/q = \textit{constant}$ with periodic boundary conditions gives

$$q = \cos(m\phi), \sin(m\phi), \quad m = 0, 1, 2, \ldots,$$

(complex form: $q = \exp(im\phi), m = 0, \pm 1, \pm 2, \ldots$)

Eigenfunctions on the surface of a sphere, cont.

With $\frac{q''}{q} = -m^2$, equation for $p(\theta)$ is

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Boundary conditions on p: insist solution is bounded at $\theta = 0, \pi$.
In the s variable, this implies $p(s = \pm 1)$ is bounded.

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- If $\lambda \neq l(l+1)$ for positive integer l, power series has radius of convergence $= 1$, and thus series diverges at $s = \pm 1$.
- For $m > 0$: remarkable formula

$$p(s) = P_l^m(s) \equiv (1 - s^2)^{m/2} \frac{d^m}{ds^m} P_l(s).$$

This requires $m \leq l$ to get nonzero answer.

Eigenfunctions on the surface of a sphere, summary

Complete set of eigenfunctions is

$$\cos(m\phi)P_l^m(\cos\theta), \quad \sin(m\phi)P_l^m(\cos\theta), \; l = 0, 1, 2, \ldots \; m = 0, 1, 2, \ldots, l$$

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These are the famous *spherical harmonics*.

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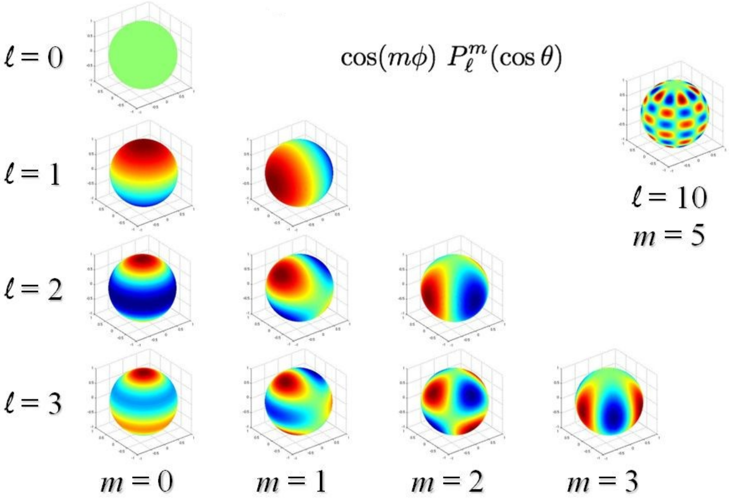
Spherical harmonics are therefore orthogonal: $< Y_l^m, Y_{l'}^{m'} >= 0$ unless $l = l'$ and $m = m'$

Table of spherical harmonics

l	m	$P_l^{\|m\|}$	Y_l^m
0	0	1	1
1	0	$s = \cos\theta$	$\cos\theta = z$
1	± 1	$(1-s^2)^{1/2} = \sin\theta$	$e^{\pm i\phi}\sin\theta = \frac{x \pm iy}{r}$
2	0	$\frac{1}{2}(3s^2-1) = \frac{1}{2}(3\cos^2\theta - 1)$	$\frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}\frac{2z^2-x^2-y^2}{r^2}$
2	± 1	$3s(1-s^2)^{1/2} = 3\cos\theta\sin\theta$	$3e^{\pm i\phi}\cos\theta\sin\theta = 3\frac{(x \pm iy)z}{r^2}$
2	± 2	$3(1-s^2) = 3\sin^2\theta$	$3e^{\pm 2i\phi}\sin^2\theta = 3\frac{(x \pm iy)^2}{r^2}$

Spherical harmonics in pictures



Example: solving Laplace equation inside sphere

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Separating variables $u = R(r)v(\phi, \theta)$ gives

$$\frac{-r^2 R'' - 2rR'}{R} = \frac{\Delta_s v}{v} = -\lambda$$

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Good news! We have solved the eigenvalue problem

$v = Y_l^m(\phi, \theta), \quad \lambda = l(l+1), \quad l = 0, 1, 2, \ldots \quad m = 0, \pm 1, \pm 2, \ldots, \pm l.$

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Now put it all together as superposition:

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_l^m(\phi, \theta), \quad A_{lm} \text{ are potentially complex.}$$

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Use orthogonality to find coefficients:

$$f(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l Y_l^m(\phi, \theta)$$

so that

$$A_{lm} = \frac{1}{a^l} \frac{\langle f, Y_l^m \rangle}{\langle Y_l^m, Y_l^m \rangle}, \quad \langle v_1, v_2 \rangle = \int_0^{\pi} \int_0^{2\pi} v_1(\phi, \theta) v_2(\theta, \phi) \sin\theta \, d\phi \, d\theta.$$

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For $l = 0$, $P = 1$ and get $Y_0^0 = 1$.
For $l = 1$, $P = x, y, z$ gives $z/r = Y_1^0$, and x/r,y/r which are the real and imaginary parts of Y_1^1.