# Questions and answers regarding my preprints on Arithmetic Teichmuller Spaces

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**Abstract**

This document puts together some questions, which I have received from several mathematicians about my recent preprints on Arithmetic Teichmuller Spaces and my work related to Mochizuki's work, along with my answers to these questions. This paper also contains a sketch of some new ideas discussed in response to Question 2.14 which will appear in [Constr. V] which is under preparation.

## Contents



## 1 Introduction

This document puts together some questions, about my work on Arithmetic Teichmuller Spaces ([Joshi, 2020a, 2022, 2023b,a,c, 2024]) on Mochizuki's work [Mochizuki, 2021a,b,c,d], which I have received from several mathematicians, along with my answers to these questions. I have not edited these questions except to add notation from my papers for clarity. In what follows, my papers Constructions I–IV in [Joshi, 2020a, 2022, 2023b,a,c, 2024] will be referred to as [Constr. I], . . ., [Constr. IV] and [Joshi, 2022] will be referred to as [Untilts paper] etc. The situation with my papers is quite fluid and evolving so readers should work with the arxiv versions and not with previous downloaded versions. This paper also contains a sketch of some new ideas regarding Kodaira-Spencer classes and relationship to the Mochizuki's construction of his classes in [Mochizuki, 1999], in response to Question 2.14 which will appear in [Constr. V] which is under preparation.

## 2 Questions and Answers

**Question 2.1.** In Thm 2.3.1 of your most recent preprint [Constr. III], why do you say that the equivalence of categories is non-canonical? Surely the equivalence is just given by the forgetful functor from the category you are considering to Sch/L (which is certainly canonical)?

I also don't understand what the proof is saying, or rather, what the relevance is of the assertions in it (which I agree with) to the statement of the Theorem

**Answer.** I agree with you of course. But I hope it is clear that in general there is no natural equivalence between two categories of schemes/arithmeticoid which is compatible with the arithmeticoids (and that is a key point here). I will change the statement and the proof in the next update (this has been done now).

**Question 2.2.** How does the failure of the absolute galois group for the Fargue-Fontaine curves play a role in all of this? I actually am not sure how the tempered fundamental groups fit into all of this either. Is this for the initial construction of the Tate parameters — you fix an untilt and then you lift them to the big rings?

**Answer.** (1) Tempered fundamental groups are needed for the theory from [IUT 1] onwards. Tempered fundamental groups [André, 2003] capture the existence of discrete, infinite coverings in rigid analytic geometry–a classic example of an important infinite discrete covering is given by Tate's construction of a Tate elliptic curve over a $p$-adic field $E$ as a rigid analytic quotient $E^*/q^{\mathbb{Z}}$ for a suitable $|q|_E < 1$. The covering $E^* \to E^*/q^{\mathbb{Z}}$ is a covering with Galois group $\mathbb{Z}$ (and an example of a Tempered covering as defined in [André, 2003]). The theory of Mumford curves provides other examples of discrete coverings in rigid analytic geometry. The tempered fundamental group is a pro-discrete group but not necessarily a pro-finite group. (2) global [Constr. II(1/2)] and local Fargues-Fontaine curves [Untilts paper] are sources (i.e. parameter spaces) for arithmetic holomorphic structures.

(a) Mochizuki's idea is to average (in a sense) over contributions from all arithmetic holomorphic structures (with chosen indeterminacies such as theta-links).

(b) complete FF curves are not uniquely determined by their anabelomorphism class (this is my thm) i.e. there are many such curves with the given top. galois group as their étale fundamental group i.e. there are many distinct moduli for arithmetic holomorphic structures.Hence Ind1 arises from the failure of the Absolute Grothendieck Conjecture.

(c) One can also think of Ind1 as symmetries of Fargues-Fontaine curves (both local and global). From Mochizuki's point of view, Ind1 comes from automorphisms of a suitable prime-strip for each prime (global case) and you can look up my approach to prime-strips in [Constr. III].

**Question 2.3.** Is there a recipe for getting log-links that are not of the perfectoid variety? Your "Constructions..." paper and your "Part II" paper seem to both be of the perfectoid persuasion. I just want to move this to the finite extension of $\mathbb{Q}_p$.

**Answer.** What you ask can't be done. According to Mochizuki [IUT 1, Page 31], the theta-link is related to Witt vectors. Once you recognize this point my construction of the theta-link in

[Constr. II(1/2)] is completely natural. Now from [IUT 1, Page 31] one knows that Mochizuki's log-link is a proxy for Frobenius. Hence after [Constr. II(1/2)] this (log-link=frobenius) becomes an exact assertion (not just as a proxy). One can relate this view of log-link to Mochizuki-style logarithm description (this is worked out in [[Constr. II(1/2)] and [Constr. III]] and is again natural).

**Question 2.4.** But then how do you prove upper semicompatibility? [IUT 3, Prop. 3.5]

That is, that your $\Psi$ region is contained in the hull of ind1,2(g*$\mathscr{I}$)?

Here $g$ is the "Gaussian" tuple of $q$'s and the $\mathscr{I}$ is locally $(1/p)\cdot\log(\mathscr{O}_K)$ but in the tensor packets of log shells.

**Answer.** One way of seeing this is through the fact that the space $H^1_e(G_E,\mathbb{Z}_p(1))$ is amphoric. Fix the standard arithmeticoid with local point $x$ (on the relevant Fargues-Fontaine curve) and fix the Bloch-Kato isomorphism of this subspace for this arithmeticoid with the integral log-shell. Now suppose $y=\varphi(x)$ for the local Frobenius then one has an isomorphism

$$H^1(G_{E,y},\mathbb{Z}_p(1))\simeq H^1(G_{E,x},\mathbb{Z}_p(1))\to(1/p)\cdot\log(\mathscr{O}_{E,x})=\text{ fixed chosen log-shell}$$

(via Bloch-Kato Isom). This is Mochizuki's upper-semi compatibility from my point of view.

**Question 2.5.** (1) It seems when you write $G_{L'_w;K_w}$ you are trying to emphasize the choice of algebraic closure that $L'_w$ sits in. First, is this the case? Second, why does this matter?

(2) I meant specifically for the Galois groups and Galois cohomology.

**Answer.** (1) Yes. In the theory (both my theory and Mochizuki's theory) the algebraic closure is dependent on the choice of the local arith. hol. str. at a prime $w$ with and importantly when arith. hol. str. moves (with various actions (in my theory) or indeterminacies in Mochizuki's theory) the alg. closure moves–to make this notion precise I use untilts which provide perfectoid alg. closed overfields plus tilting data–which can be distinguished from each other (i.e. untilts are isomorphic or not isomorphic) and which can be treated via Fargues-Fontaine theory.

(2) Yes again: specific alg. closure provides (a) absolute galois group $G$ (b) roots of unity on which $G$ operates and this determines a specific instance of each of the Galois cohomology groups $H^1(G,\mathbb{Z}(1))$, $H^1(G,\mathbb{Z}_p(1))$ etc. The isomorphism class of these groups is independent of the choice of the alg. closure of course.

In Mochizuki's terminology $\mathbb{Z}(1),\mathbb{Z}_p(1)$ equipped with $G$-action are cyclotomes and so each choice of arith. hol.str. provides a choice of these cyclotomes and all these choices are abstract isomorphism but are attached to distinct arithmetic holomorphic structures

**Question 2.6.** Are you just looking at the underlying topological space. Is that the point?

**Answer.** No. Berkovich analytic spaces. The structure of analytic or holomorphic function spaces is of paramount importance in my theory and that is why I have asserted that the theory looks like classical Teichmuller Theory. Mochizuki works with (fundamental) groups arising from these spaces.

**Question 2.7.** What is the affect of considering *all topological isomorphisms* in your construction of $\Psi$ in the context of [Constr. II(1/2), Proposition 5.7.1]?

**Answer.** I should be a bit more precise: I want all isomorphisms of the topological $\mathbb{Z}_p$-module $H^1(G_E, \mathbb{Z}_p(1))$ induced by isomorphisms of the pair ($G_E$ acting on $\mathbb{Z}_p(1)$). Using all such isomorphisms is the analog of "polyisoms" in [IUT 1-4]. This is necessary to avoid logical fallacies. For example we could have chosen (by intent or by accident) a rather special isomorphism which could mean that someone could arrive at a different bound (or inequality) by working with a different isomorphism.

**Question 2.8.** I mean that there is a set $\Omega$ which is defined in [Dupuy-Hilado] paper and alluded to in Mochizuki (you take the region of $-P_{\Theta}$, apply ind3 (this is in the log-Kummer manuscript of [Dupuy-Hilado]) and then apply ind1 and ind2.

You have a region before the hull which is your $\widetilde{\Theta}^{Mochizuki}$. Let's just call this $J$.

I want to, using your apparatus, to know if $\Omega$ is a subset of $J$ and $J$ is a subset of $\Omega$.

**Answer.** Regarding your question of the relationship between my theta-values set and Mochziuki's set. In my opinion, Mochizuki's construction of his set needs work because of the issue of the existence of arithmetic holomorphic structures. That being said, the key point is making precise how various indeterminacies arise–for non-trivial reasons–this is the central issue faced by [Scholze and Stix, 2018]. This is done correctly and rigorously in my approach via construction of arithmetic holomorphic structures and the construction of Mochizuki's Ansatz (which is the set of all possible $\Theta_{gau}$-Links) in [Constr. III] (Rosetta Stone (of that paper) and Paragraph 8.12 provide a way of transliterating my results to Mochizuki's context). This is the key from my point of view.

Once this is done, one obtains a set which is stable under action of Galois (which moves arithmetic hol. strs.) and also under global Frobenius (for Mochizuki this is working with a vertical column of log-links). This allows for Ind2, Ind3 (by my work one has a natural interpretation of indeterminacies). In this approach Ind1 appears via failure of the Absolute Grothendieck Conjecture for Fargues-Fontaine curves (proved by me in [Joshi, 2020b]) and one makes its effect count via Galois enlargements (as defined in [Constr. III]).

**Question 2.9.** Why are tempered fundamental groups involved in the definition of (local) arithmetic holomorphic structures.

[Just as a remark I am trying to formally trace through the construction of the region from start to end. I want to see what automorphisms of these fundamental groups are doing and see what exactly the anabelian input is.]

**Answer.** The anabelian input (for IUT 1-4) is a bit of a red herring in my opinion. This idea has its origin in [Mochizuki, 2010] (or even before that) where passage to coverings suggests that the theory is necessarily anabelian (i.e. group theoretic) in origin. [On the other hand, passage to (abelian) coverings is already used in Diophantine geometry–this goes back to Parshin and Arakelov (and also exploited by Faltings). So covering constructions do not signify that theory is anabelian in origin (in my opinion). [Grothendieck's section conjecture for finite extensions of $\mathbb{Q}$ is of course Diophantine and Anabelian in nature. But section conjecture is not proved and it does not feature in [IUT 1-4].]

From my point view primary objects for both the theories are spaces: groups arise from spaces. [Mochizuki does not work with spaces.] Once one works with spaces one gains much clarity about the theory.

To some extent my work has deconstructed the anabelian point of view.

But the following point will be useful to understand. What I observe is that there is a local system in Galois groups over Fargues-Fontaine curve. This local system has fiber $G_{E,K_x}$ as a closed classical point $x$ with the Galois group being computed with the algebraic closure provided by the residue field $K_x$ of $x$ i.e. one has "bundle of groups" over the complete curve whose fiber at a closed classical point $x$ is $G_{E,K_x}$. Similar construction can be carried out over the incomplete curve. A similar local system in tempered fundamental groups has fiber $\Pi^{temp}_{X/E;K_x}$. Mochizuki clearly wanted to work with these types of local systems. My approach to these is the broader idea of "anabelian variations of $\Pi$" introduced in [Joshi, 2021]. This should be viewed as proxy for a variation of Hodge structures in the anabelian setting. To make this precise one should work with Scholze's Theory of Diamonds.

In the topological context this is a familiar object over the fundamental groupoid and bundle of groups (see Bundle of Groups on Wikipedia for instance). This is the sort of thing Mochizuki wanted to do. [This remark will be added in the next update of the [Untilts].]

**Question 2.10.** In the context of [Constr. II(1/2)], your $\tilde{K}$ is multiplicative monoid, but why are you identifying it with multiplicative structure of the field?

It is unclear to me whether the absolute value is changing or the additive structure is changing. In the archimedean case, the first one is changing.

Can you (Mochizuchi?) be more explicit about actual deformations?

**Answer.** (1) The precise assertion which you are referring to is that there exists an isomorphism of multiplicative monoids $\tilde{K} \simeq K^\flat$. [This is how the multiplicative structure of $K$ can be related to that of its tilt. There is no other relationship.] The perfectoid field $K$ (with tilt $\mathbb{C}_p^\flat$) is a valued field and the multiplicative structure of $K^* \simeq$ the value-group $p^{\mathbb{Q}} \times \mathscr{O}_K^*$ and the tilde version of the unit group portion is also identified with the unit group portion of the tilt.

(2) The kernel of $\log : \tilde{K} \to K$ is a one-dimensional $\mathbb{Q}_p$ vector space which is isomorphic with the Tate module of the multiplicative group tensored with $\mathbb{Q}_p$ and $\tilde{K}$ modulo this one dimensional $\mathbb{Q}_p$ vector space is isomorphic to the additive group $K$.

Mochizuki works with $\log : \bar{\mathbb{Q}}_p^* \to \bar{\mathbb{Q}}_p$ and the exact sequence this gives rise to. [Mochizuki's idea of working with $\bar{\mathbb{Q}}_p$ and $\log$ is problematic in analytic contexts because working with log requires convergence in $\bar{\mathbb{Q}}_p$ but this is not complete. So Mochizuki tries to manage this issue with direct limits and so on. I think this is just bad from $p$-adic analytic point of view and proving anything is quite nightmarish and unclear to me.]

(3) Mochizuki also needs to manipulate the value group for defining his theta-link. This is exactly what my approach to the theta-link does in [Constr. II] (global case is in [Constr. III]). In my approach value group type manipulations are done via the tilt $K^\flat$ which is isomorphic to $\mathbb{C}_p^\flat$. This is one of the reasons why one needs to work with untilts rather than with perfectoid fields. Second reason being computing arithmetic degrees requires working in a fixed value group

(which I get by fixing the tilt). [Mochizuki needs a fixed value group in the proof of Corollary 3.12 and he has not really described how to deal with them carefully enough that is why his Cor. 3.12 is problematic.] There is another reason why this is problematic: one wants this sequence to move in its isomorphism class as the arith. hol. str. moves. This point is difficult to quantify from Mochizuki's point of view but not from my point using untilts. Each untilt provides its own version of the log-sequence!

(3) Via the logarithm, the multiplicative structure can be related to the additive structure so one can describe the theory in terms of additive formal groups instead. But one must decouple multiplication (or its additive avatar) from the field structure (because the field structures need not be isomorphic) to see the deformations. Mochizuki's way of asserting this is multiplication and addition are two separate dimensions and one must decouple the two to see his theory [I am saying that this is a correct assertion and I can prove it via Fargues-Fontaine...]

The deformation of the number field is a choice of an arithmeticoid which gives an adeloid and adeloid + embedding of $L$ in it provides essentially a version of the global component of Mochizuki's Hodge Theater (in Mochizuki's case this is a realified Frobenioid of the number field).

(4) Global frobenius I construct moves arithmetic hol. struct. by moving points on the Fargues-Fontaine curves at all $v$. There is also an action of any given alg. number $0 \neq x \in L$ which moves arithmetic holomorphic structures at finitely many primes in the support of the fractional ideal $x$ by powers of Frobenius at such primes dependent on their multiplicities in $(x)$. In [IUT 1-3] moving by global Frobenius corresponds to moving in the vertical column of log-links (by Constructions II appendix) and what it accomplishes is that absolute values of the Tate parameter grow (this I prove in Constructions II appendix).

**Question 2.11.** As to your current papers, I think it would be extremely helpful if you could give a concise explanation of how the global arguments you allude to eventually feed into a bound for the sum of local quantities. It would be ideal to have a 'plausibility argument' expressed in mostly For example, in the function field case, we know that the KS map gives us a non-trivial map from (1/6)-th of the discriminant sheaf to the log differentials. If there were a plausibility argument of something approaching that level of simplicity, it would be really great.

**Answer.** There several questions posed here.

(1) As to the factor 1/6: it appears in Mochizuki's proof through [Constr. IV, Prop. 5.6.1] i.e. through standard and well-understood reasons stemming from relationships between various height functions.

(2) Existence of Initial Theta Data [Constr. IV, Theorem 5.7.1] is a global argument. There are other global points too–for example working with a compactly bounded subset in [Constr. IV] or [IUT 4] is another example of a global datum which needs to be fixed in some of the estimates. Working with global normalized arithemeticoids means each arithmeticoid provides its own private version of the Product Formula (for a given fixed number field) for normalized valuations. But one cannot normalize all the arithmeticoids simultaneously [Constr. II(1/2), Section 5].

(3) Mochizuki does not approach Szpiro directly. He proves Vojta's Inequality and there is no geometric version of this as far as I can tell. So what you ask does not quite exist. But let me make the following remarks.

It will be useful to read my approach to the Bogomolov, Zhang proofs [Constr. II(1/2), Section 10] where I show (complementing Mochizuki's own comparison of his proof with these proofs) how all the structures (theta-links, log-links, and global Frobenius morphism) come into play in their proofs. In the Bogomolov, Zhang proofs, the geometric Szpiro inequality emerges from two global facts (a) the well-known relation between generators of fundamental group of a Riemann surface (b) global Frobenius and its relationship with the height function ("global Frobenius allows us to Fudge heights").

In Mochizuki's proof the two global facts are (a') Product formula (b') existence of global Frobenius, whose existence I prove (in [Constr. II(1/2), Section 4.2]) and which appears as Mochizuki's log-link and is a proxy for global Frobenius [IUT3, Page 5-7] or [IUT3 Printed version, Pages 409-411] which allows one to "fudge" heights (or heights are fungible up to global Frobenius). My discussion of this process with examples is in [Constr. II(1/2), Section 8, especially Remark 8.5.2]. Neither Mochzuki nor I have the full theory worked out (from the point of view espoused in [Constr. II(1/2)]) but all the key properties asserted by Mochizuki in this context are established in [Constr. II(1/2)] and [Constr. II].

(3) Important point in [IUT 4] is that Mochizuki proves inequalities over compactly bounded subsets of moduli. Notably he does not work with the whole of moduli. The reason for working with compactly bounded subsets is because of Mochzuki's reduction [Mochizuki, Theorem 2.1] This is an important point which distinguishes previous works (such as [Kim, 1997], [Beauville, 2002]) on geometric Szpiro Inequality. Notably again, Mochizuki [IUT 4] does not directly prove Szpiro. But proves Vojta's Inequality for suitable compactly bounded subsets which implies the general case of Vojta's Inequality which implies abc (by Frankenhuysen's theorem). As is well-known Vojta's Inequality does not have a nice geometric formulation.

I will address your question about Kodaira-Spencer classes as a separate question as this is an important aspect of my theory.

**Question 2.12.** Isn't your [Joshi, 2021, Theorem 2.15.1] true without the hypothesis that

$$K_1^\flat \simeq K_2^\flat.$$

**Answer.** That is correct. Any pair $K_1, K_2$ of non-isomorphic, alg. closed, complete valued fields each $K_1, K_2$ containing an isometrically embedded $\mathbb{Q}_p$ can be used in [Joshi, 2021, Theorem 2.15.1].

**Question 2.13.** Then what is the purpose of fixing the tilt in [Constr. I, Theorem 2.15.1]?

**Answer.** Fixing the tilt to $F = \mathbb{C}_p^\flat$ or some alg. closed, perfectoid $F$ of char. $p > 0$ allows us to fix the unit group $1 + \mathfrak{m}_F$. Because of the close relationship ([Fargues and Fontaine, 2018, Définition 2.3.1]) this unit group has to the Fargues-Fontaine curve $\mathscr{Y}_{F,\mathbb{Q}_p}$, means that $\sigma \in \mathrm{Aut}_{\mathbb{Z}_p}(1 + \mathfrak{m}_F)$ is identified with the (precise form of) Mochizuki's Indeterminacy Ind2 and one sees immediately how $\sigma$ moves arithmetic hol. strs. We can assert this relationship with Mochizuki's Ind2 because this unit group is perfect in Mochizuki's sense–i.e. it is a $\mathbb{Q}$-vector space (which follows from [Fargues and Fontaine, 2018] as this unit group is a $\mathbb{Q}_p$-Banach space).

{qdr1}

**Question 2.14.** Now to the issue of Kodaira-Spencer classes: are there any here?

**Answer.** Kodaira-Spencer Theory is the subject of [Constr. V] which is in preparation. So this is just a very rough sketch.

Locally, at each prime [Constr. I, Theorem 2.15.1] may be viewed as a classical deformation theoretic statement in the sense of deformation of complex structures.

Furthermore it is a consequence of an old observation of Berkovich that under suitably strong form of resolution in char. $p > 0$, the underlying topological spaces of the rigid analytic spaces occurring in [Constr. I, Theorem 2.15.1] are of the same homotopy type (I will state this in the next update of [Constr. I]). Specifically, Berkovich's assertion is that the homotopy type of the analytic space obtained from a reasonable variety over an algebraically closed, complete valued field is invariant under further base change to a bigger algebraically closed complete valued field. (for curves this homotopy assertion is valid by Mochizuki's work) and for this case the relevant resolution is available.

So the theory of my papers is almost exactly as in the classical situation: analytic structures are distinct, underlying top. spaces are homotopic (at least conjecturally–subject to resolution in char. $p > 0$) and tempered fundamental groups are isomorphic.

So Kodaira-Spencer classes are not unexpected. A formal treatment of the ideas sketched here will appear in [Constr. V] which is under preparation. [This is work in progress and I request anyone using my ideas to attribute them to me.]

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