TWISTED LINEAR PERIODS AND A NEW RELATIVE TRACE FORMULA

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ABSTRACT. We study the linear periods on GL_{2n} twisted by a character using a new relative trace formula. We establish the relative fundamental lemma and the transfer of orbital integrals. Together with the spectral isolation technique of Beuzart-Plessis–Liu–Zhang–Zhu, we are able to compare the elliptic part of the relative trace formulae and to obtain new results generalizing Waldspurger's theorem in the $n = 1$ case.

CONTENTS

1. INTRODUCTION

1.1. Linear periods.

Let F be a number field and E a quadratic field extension of F, with their rings of adèles denoted by $\mathbb{A}_F$ and $\mathbb{A}_E$. Let $\eta : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$ be the quadratic character attached to E/F by class field theory. Let $\omega : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ and $\chi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ be two characters with $\chi^n|_{\mathbb{A}_F^\times} \omega = 1$. Let $\mathcal{A}$ be a central simple algebra over F of dimension $4n^2$. Fix an embedding $E \to \mathcal{A}$ of F-algebras, and let $\mathcal{B}$ be the centralizer of E in $\mathcal{A}$. Let $G = \mathcal{A}^\times$ and $H = \mathcal{B}^\times$, both regarded as algebraic groups over F. Let Z be the center of G. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ with central character ω. Take $\varphi \in \pi$ and define

$$(1.1) \qquad P_\chi(\varphi) = \int_{Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \varphi(h)\chi(h)\mathrm{d}h.$$

We propose the following conjecture.

Conjecture 1.1. *(i) If P_χ is not identically zero, then $L(\frac{1}{2}, \pi_{0,E} \otimes \chi) \neq 0$ and $L(s, \pi_0, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times})$ has a simple pole at $s = 1$ where π_0 is the Jacquet–Langlands transfer of π to $\mathrm{GL}_{2n}(\mathbb{A}_F)$, and $\pi_{0,E}$ is the base change of π_0 to $\mathrm{GL}_{2n}(\mathbb{A}_E)$. Moreover for all places v of F, the Langlands parameter of $\pi_{0,v}$ takes value in $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude factor $\chi_v|_{F_v^\times}$, and $\epsilon(\pi_{0,E,v} \otimes \chi_v)\eta_v(-1)^n\chi_v(-1)^n = (-1)^r$ if $v \in S$, where r is the split rank of G over F_v. This integer r is also the integer such that $\mathcal{A}_v = \mathcal{A} \otimes F_v = M_r(C)$ where C is a division algebra over F_v.*

(ii) If all archimedean places of F split in E, or n is odd, we also have a converse. Let π_0 be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. Assume that $L(\frac{1}{2}, \pi_{0,E} \otimes \chi) \neq 0$ and $L(s, \pi_0, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times})$ has a simple pole at $s = 1$. Then there is a central simple algebra $\mathcal{A}$ containing E over F and an irreducible cuspidal representation π of $G(\mathbb{A}_F)$, such that π is the Jacquet–Langlands transfer of π_0 and P_χ is not identically zero on π. Moreover if n is odd, we can take $\mathcal{A}$ to be a matrix algebra over a (possibly split) quaternion algebra.

If $n = 1$, then this is the celebrated result of Waldspurger [Wal85]. If χ is trivial and $\mathcal{A}$ is of the form $M_n(D)$ where D is a quaternion algebra over F containing E, the conjecture reduces to the one proposed by Guo and Jacquet in [Guo96]. The description of the split rank of G over F_v and the root number of π_v, regardless of χ being trivial or not, is (a consequence of) the conjecture of Prasad and Takloo-Bighash [PTB11, Conjecture 1].

When χ is trivial, a relative trace formula approach was proposed in [Guo96], generalizing the work of [Jac86]. The study of these relative trace formulae yields both local and global results towards Conjecture 1.1 in the case χ being trivial, cf. [FMW18, Li21, Li22, Xue21]. However these relative trace formulae make essential use of the fact that $L(s, \pi_{0,E})$ factors, i.e. $L(s, \pi_{0,E}) = L(s, \pi_0)L(s, \pi_0 \otimes \eta_{E/F})$, so they cannot be extended to the case of nontrivial χ.

The goal of this paper is to propose a new relative trace formula towards Conjecture 1.1, and to confirm many cases of the conjecture by comparing the elliptic parts of the relative trace formula. We do this by establishing the relevant fundamental lemma and transfer of orbital integrals.

Remark 1.2. The assumption that E/F splits at the archimedean places comes from the fact that the only central simple algebras over $\mathbb{R}$ are matrix algebras over quaternion algebras. In the case n being even, they do not provide us with enough orbits in our relative trace formula approach. A similar phenomenon appears even when χ is trivial, cf. [Guo96].

Remark 1.3. The statement of Conjecture 1.1 also makes sense when $E = F \times F$ provided that the embedding $E \to \mathcal{A}$ makes $\mathcal{A}$ a free E-module. If χ is trivial, some preliminary studies have been carried out in [Zha15b]. The trace formula we proposed in this paper, with obvious modifications, can be used to study the general case. We will return to this in the future work.

For simplicity we assume for the rest of this paper that $\mathcal{A} = M_n(D)$ where D is a (possibly split) quaternion algebra over F containing E. Thus $G = \mathrm{GL}_n(D)$ and $H = \mathrm{Res}_{E/F}\mathrm{GL}_{n,E}$.

Theorem 1.4. *Assume that E/F is split at all archimedean places, $\pi_{0,E}$ is cuspidal, and there is at least one place v_1 such that π_{v_1} is elliptic. Then part (i) of Conjecture 1.1 holds.*

Here the ellipticity is relative to the subgroup H and its precise meaning can be found in Subsection 3.3. By Proposition 3.4, all supercuspidal representations are elliptic.

The local part of this theorem, i.e. the self-duality of π and determining the local root numbers, confirms the conjecture of Prasad and Takloo-Bighash in many cases. The general case of the conjecture will be treated in a subsequent paper, based on the results we obtain here.

In the converse direction, because of the lack of certain representation theoretic results, our theorem is less general than the above one.

Theorem 1.5. *Assume that n is odd, E/F is split at all archimedean places and that π_0 satisfy the conditions in part (ii) of Conjecture 1.1. Assume $\pi_{0,E}$ is cuspidal. Let Σ be a finite set of finite places of F containing dyadic places, such that if $v \notin \Sigma$, then E_v/F_v, $\pi_{0,v}$ and χ_v are all unramified. Assume that*

(1) *if $v \in \Sigma$, then either v splits in E or π_{0,E_v} is supercuspidal.*
(2) *there is at least one place v_1 in Σ such that $\pi_{0,E_{v_1}}$ is elliptic.*

Then there is a unique quaternion algebra D over F and an irreducible cuspidal representation π of $G(\mathbb{A}_F)$, such that π is the Jacquet–Langlans transfer of π_0 and P_χ is not identically zero on π, i.e. part (ii) of Conjecture 1.1 holds.

Here the ellipticity is relative to subgroups of $\mathrm{GL}_{2n}(E_{v_1})$ and the precise meaning will be explained in Subsection 3.4.

1.2. **The new relative trace formula.** Let v be an archimedean place of F, we let $\mathcal{S}(G(F_v))$ be the space of Schwartz functions on $G(F_v)$, meaning the functions f such that Df is bounded on $G(F_v)$ for all algebraic differential operators D on $G(F_v)$. Let $\mathcal{S}(G(\mathbb{A}_F))$ be the space of Schwartz functions on $G(\mathbb{A}_F)$, i.e. linear combinations of the functions of the form $\prod_v f_v$ where $f_v \in C_c^\infty(G(F_v))$ if v is nonarchimedean and $f_v \in \mathcal{S}(G(F_v))$ is v is archimedean. Similar definitions also applies to other groups.

To study the linear period P_χ we consider the following relative trace formulae. Let $f \in \mathcal{S}(G(\mathbb{A}_F))$. We put

$$K_f(g_1, g_2) = \int_{Z(F)\backslash Z(\mathbb{A}_F)} \sum_{y \in G(F)} f(zg_1^{-1}yg_2)\omega(z)^{-1}\mathrm{d}z.$$

Define a distribution

(1.2) $$J(f) = \iint_{(Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F))^2} K_f(h_1, h_2)\chi(h_1h_2^{-1})\mathrm{d}h_1\mathrm{d}h_2.$$

This distribution at least formally unfolds geometrically and has a spectral expansion. Then we obtain a relative trace formula on $G(\mathbb{A}_F)$. This relative trace formula is essentially the same as the one propose in [Guo96], except that a character χ is inserted.

3

We now propose a relative trace formula to study the L-function $L(\frac{1}{2}, \pi_{0,E} \otimes \chi)$. This is the main innovation of this paper.

Let us first recall the relative trace formula proposed by [Guo96] when χ is trivial. In this case, let $f' \in \mathcal{S}(\mathrm{GL}_{2n}(\mathbb{A}_F))$ and the relative trace formula results from the geometric and spectral expansion of the distribution

$$\iint \sum_{x \in \mathrm{GL}_{2n}(F)} f'(h_1^{-1} x h_2) \eta(h_1) \mathrm{d}h_1 \mathrm{d}h_2,$$

where the integration is over $h_1, h_2 \in \mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F)$. The spectral expansion gives both the periods

$$\int \varphi(h) \mathrm{d}h, \quad \int \varphi(h) \eta(h) \mathrm{d}h,$$

where $\varphi \in \pi_0$ and the domain of the integration in both cases are $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F)$. Thus by the work of Friedberg and Jacquet [FJ93] on (split) linear periods these periods give rise to the L-functions $L(s, \pi_0) L(s, \pi_0 \otimes \eta)$.

It is clear that such an approach cannot be generalized to arbitrary χ, simply because $L(s, \pi_{0,E} \otimes \chi)$ does not factorize in general. An alternative approach is needed. Assume the central character of π_0 is ω. In the case of $n = 1$, Jacquet [Jac87] proposed the following. Assume $n = 1$. Let $f' \in \mathcal{S}(\mathrm{GL}_2(\mathbb{A}_E))$ be a test function. Then consider

$$\iint \sum_{x \in \mathrm{GL}_2(E)} f'(h_1^{-1} x h_2) \chi(h_1)(\omega \eta)(\lambda(h_2)) \mathrm{d}h_1 \mathrm{d}h_2.$$

Here the integration is over $h_1 = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$, $a \in E^{\times} \backslash \mathbb{A}_E^{\times}$, h_2 is in $\mathrm{GU}(1,1)$ where $\mathrm{GU}(1,1)$ stands for the quasisplit similitude unitary group in two variables, and λ is the similitude character. Jacquet's idea is as follows. The integration over h_1 gives the central L-value, and the period over $\mathrm{GU}(1,1)$ ensures that the representations we consider on $\mathrm{GL}_2(\mathbb{A}_E)$ are all base change from $\mathrm{GL}_2(\mathbb{A}_F)$. Jacquet (re)proved Conjecture 1.1 in the case $n = 1$ based on this relative trace formula.

Thus for general n a natural idea is to extend the relative trace formulae in [Jac87], i.e.

(1.3) $$\iint \sum_{x \in \mathrm{GL}_{2n}(E)} f'(h_1^{-1} x h_2) \widetilde{\chi}(h_1)(\omega \eta)(\lambda(h_2)) \mathrm{d}h_1 \mathrm{d}h_2,$$

with $h_1 = (h_{11}, h_{12}) \in \mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_E)$, $\widetilde{\chi}(h_1) = \chi(h_{11} h_{12}^{-1})$, and $h_2 \in \mathrm{GU}(n,n)(\mathbb{A}_F)$. This is very natural and was indeed our first attempt. But it does not seem to be the correct approach and we eventually abandoned it for the following reason. The stabilizers in the geometric expansions of the distributions (1.2) and (1.3) are very different, one being tori in $\mathrm{GL}_n(F)$ and the other being tori in the unitary groups. On the philosophical level it is not expected that two trace formulae can be compared unless the stabilizers from their geometric side are closely related, e.g. they are isomorphic or at least one of them is trivial. After all, in the comparison of the trace formulae,

we need to equate the volume of these stabilizers. Therefore we do not expect a nice comparison between the geometric expansions of the distributions (1.2) and (1.3).

We take an alternative approach in this paper. The starting point is the following key observation. Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ and $\varphi \in \Pi$. The integration of φ over $\mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_E)$ does not merely tell us something about $L(\frac{1}{2}, \Pi \otimes \chi)$, but also about the self-duality of Π. Let us introduction some notation. Let $H' = \mathrm{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_n)$, and $\chi_{H'}$ is a character of H' sending $h_1 = (h_{11}, h_{12})$ to $\chi(h_{11}\overline{h_{12}})$. Indeed if

$$\int \varphi(h_1)\chi_{H'}(h_1)\mathrm{d}h_1 \neq 0$$

where the integration is over H', by [FJ93] we have $L(\frac{1}{2}, \Pi \otimes \chi) \neq 0$ and $L(s, \Pi, \wedge^2 \otimes \chi\chi^c)$ has a pole at $s = 1$ where $\chi^c(g) = \chi(\overline{g})$. The later implies that $\Pi^\vee \simeq \Pi \otimes \chi\chi^c$. What we need in the relative trace formula is to use the second integral over h_2 to separate those Π with $\Pi \simeq \Pi^c$. Under the condition that $\Pi^\vee \simeq \Pi \otimes \chi\chi^c$, this is equivalent to $\Pi^\vee \otimes \chi^{-1} \simeq \Pi^c \otimes \chi^c$ where $\Pi^c(g) = \Pi(\overline{g})$, and this later condition can be detected using the period integral of Flicker and Rallis, i.e. the integration

$$\int \varphi(h_2)\chi\eta(h_2)\mathrm{d}h_2$$

where $h_2 \in \mathrm{GL}_{2n}(\mathbb{A}_F)$. Thus our new distribution on $\mathrm{GL}_{2n}(\mathbb{A}_E)$ reads the following

$$\iint \sum_{x \in \mathrm{GL}_{2n}(E)} f'(h_1^{-1}xh_2)\chi_{H'}(h_1)(\chi\eta)^{-1}(h_2)\mathrm{d}h_1\mathrm{d}h_2,$$

where $h_1 \in \mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_E)$ and $h_2 \in \mathrm{GL}_{2n}(\mathbb{A}_F)$. The geometric and spectral expansions of this distribution give the relative trace formula on $\mathrm{GL}_{2n}(\mathbb{A}_E)$. The stabilizer of any (relatively) regular semisimple orbit is a torus in $\mathrm{GL}_n(F)$ of the form $\prod_i \mathrm{Res}_{F_i/F} \mathrm{GL}_1$, and hence it matches the stabilizers arising from the distribution (1.2).

The majority of this paper compares the elliptic part of this relative trace formula with the one on $G(\mathbb{A}_F)$. Our key local results are the relevant fundamental lemma and transfer of orbital integrals. With the recent technique from [BPLZZ21] to isolate cuspidal spectra, these local results lead to the main theorems. To remove the unnecessary conditions in those theorems, one would need to compare the full relative trace formulae, not just the elliptic part. Nevertheless the current comparison is sufficient for the purpose of solving the local problems, i.e. the conjecture of Prasad and Takloo-Bighash. The conjecture of Prasad and Takloo-Bighash in turn appears to be an indispensable ingredient in the comparison of the full relative trace formulae. We hope to address these questions in a future work.

Let us end the discussion by mentioning that the work of Getz and Wambach [GW14, p. 5–6] speculates a general principle which suggests a comparison of relative trace formulae for period integrals along symmetric subgroups. Our new relative trace formulae, apart from the characters,

are compatible with this general principle. In the notation of [GW14], their H is $\mathrm{GL}_{n,F}$, G is $\mathrm{GL}_{n,E}$ and the involution θ is the conjugation by $\begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}$.

1.3. Notation and Convention.
Throughout this paper we keep the following notation and convention.

If X is a set, we denote by $\mathbf{1}_X$ the characteristic function of it.

If G is a group and f is a function on G then we put $f^\vee(g) = f(g^{-1})$.

When a group A acts on a set X and $x \in X$ we always denote by A_x the stabilizer of x in A.

The $n \times n$ identity matrix is denoted by 1_n, or simply 1 when the size of the matrix is clear.

If F is a number field, we put $F_\infty = \prod_{v|\infty} F_v$.

Let E/F is be quadratic field extension. The nontrivial Galois involution is denoted by $\bar{\cdot}$. By a twisted conjugation by $g \in \mathrm{GL}_n(E)$, we mean the map $x \mapsto gx\bar{g}^{-1}$. The stabilizer of x in $\mathrm{GL}_{n,E}$ under this twisted conjugation is denoted by $(\mathrm{GL}_{n,E})_{x,\text{twisted}}$. This is an algebraic group over F and

$$(\mathrm{GL}_{n,E})_{x,\text{twisted}}(F) = \mathrm{GL}_n(E)_{x,\text{twisted}} = \{g \in \mathrm{GL}_n(E) \mid gx\bar{g}^{-1} = x\}.$$

We define $N : \mathrm{GL}_n(E) \to \mathrm{GL}_n(E)$ the norm map $Ng = g\bar{g}$. The image of the norm map is denoted by $N\,\mathrm{GL}_n(E)$.

Let D be a quaternion algebra over F with a fixed embedding $E \to D$. We fix an element $\epsilon \in NE^\times$ or $\epsilon \in F^\times \backslash NE^\times$ depending on whether D split or ramifies. The group $\mathrm{GL}_n(D)$ is realized as a subgroup of $\mathrm{GL}_{2n}(E)$ consisting of elements of the form

$$\begin{pmatrix} A & \epsilon B \\ B & A \end{pmatrix}, \quad A, B \in M_n(E),$$

We let $\theta : \mathrm{GL}_{2n}(E) \to \mathrm{GL}_{2n}(E)$ be the involution

$$g \mapsto \theta(g) = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

2. Relative trace formulae: the geometric side

2.1. Geometric side.
Let E/F be a quadratic extension of number fields, and D a (possibly split) quaternion algebra over F containing E. Let $G = \mathrm{GL}_n(D)$, $Z = \mathrm{GL}_{1,F}$ the center of G, and $H = \mathrm{Res}_{E/F}\,\mathrm{GL}_{n,E}$.

Let us recall from the introduction that we have the following relative trace formula on $G(\mathbb{A}_F)$. Let $f \in \mathcal{S}(G(\mathbb{A}_F))$. We put

$$K_f(g_1, g_2) = \int_{Z(F)\backslash Z(\mathbb{A}_F)} \sum_{y \in G(F)} f(zg_1^{-1}yg_2)\omega(z)^{-1}\mathrm{d}z.$$

Define a distribution

$$J(f) = \iint_{(Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F))^2} K_f(h_1, h_2)\chi(h_1 h_2^{-1})\mathrm{d}h_1 \mathrm{d}h_2.$$

We consider the $H \times H$ action on G by $(h_1, h_2) \cdot y = h_1^{-1} y h_2$. An element $y \in G(F)$ is called regular semisimple if the stabilizer $(H \times H)_y$ is a torus of dimension n over F. It is called elliptic if in addition that this torus is anisotropic modulo the center of G. Let $G(F)_{\mathrm{reg}}$ and $G(F)_{\mathrm{ell}}$ be the subsets of regular semisimple and elliptic elements. These definitions also apply to elements in $G(F_v)$ where v is a place of F.

Assume that $f = \otimes f_v$ is decomposable and there is one nonsplit place v_1 of F such that f_{v_1} is supported in the regular elliptic locus. Then we have

(2.1) $$J(f) = \sum_{y \in H(F)\backslash G(F)_{\mathrm{ell}}/H(F)} \mathrm{vol}((H \times H)_y)O^G(y, f),$$

where

$$O^G(y, f) = \int_{(H\times H)_y(\mathbb{A}_F)\backslash (H\times H)(\mathbb{A}_F)} f(h_1^{-1} y h_2)\chi(h_1^{-1} h_2)^{-1}\mathrm{d}h_1 \mathrm{d}h_2.$$

In these expressions we fix compatible measures on $Z(\mathbb{A}_F)\backslash (H \times H)_y(\mathbb{A}_F)$, $Z(\mathbb{A}_F)\backslash (H \times H)(\mathbb{A}_F)$ and $(H \times H)_y(\mathbb{A}_F)\backslash (H \times H)(\mathbb{A}_F)$ for each $y \in G(F)_{\mathrm{ell}}$ and $\mathrm{vol}((H \times H)_y)$ stands for the volume of

$$Z(\mathbb{A}_F)(H \times H)_y(F)\backslash (H \times H)(\mathbb{A}_F).$$

This integral is absolutely convergent for all regular semisimple $y \in G(F)$. Since the test function f is not compactly supported, the absolute convergence of (2.1) needs explanation. This will be given in Appendix A.

For any place v of F, we define similarly the local orbital integrals, except we integrate over $(H \times H)(F_v)$ instead.

The orbital integral can be simplified as follows. If $g \in \mathrm{GL}_{2n}$ we define an involution

$$\theta(g) = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

Then H is the group of fixed point of θ. We introduce the symmetric space

$$S = \{g\theta(g)^{-1} \mid g \in G\}$$

and then H acts on S by conjugation. Put

$$\widetilde{f}(g\theta(g)^{-1}) = \int_{H(\mathbb{A}_F)} f(gh)\chi(gh)^{-1}\mathrm{d}h.$$

Then $\widetilde{f} \in \mathcal{S}(S(\mathbb{A}_F))$ and

$$O^G(g, f) = O^S(s, \widetilde{f}) = \int_{H_s(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \widetilde{f}(h^{-1} sh)\mathrm{d}h, \quad s = g\theta(g)^{-1}.$$

For any place v of F, the local orbital integrals can be simplified in a similar way.

7

We introduce another relative trace formula with which (2.1) will be compared. While the distribution J does not differ much from those in [Guo96], this relative trace formula is the main point of innovation of the present paper.

Let $G' = \operatorname{Res}_{E/F} \operatorname{GL}_{2n}$, $H' = \operatorname{Res}_{E/F}(\operatorname{GL}_n \times \operatorname{GL}_n)$ embedded in G' as diagonal blocks, and $H'' = \operatorname{GL}_{2n,F}$. Let $Z' \simeq \operatorname{GL}_{1,F}$ embedded in G' diagonally. Let $f' \in \mathcal{S}(G'(\mathbb{A}_E))$ and

$$K_{f'}(g_1, g_2) = \int_{Z'(E)\backslash Z'(\mathbb{A}_E)} \sum_{\gamma \in G'(F)} f'(g_1^{-1} z \gamma g_2) \omega(z\bar{z})^{-1} \mathrm{d}z.$$

For $h = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \in H'(\mathbb{A}_F)$ with $h_1, h_2 \in (\operatorname{Res}_{E/F} \operatorname{GL}_n)(\mathbb{A}_F)$, we put $\chi_{H'}(h) = \chi(h_1\overline{h_2})$. Consider the distribution

$$I(f') = \int_{Z'(\mathbb{A}_E)H'(F)\backslash H'(\mathbb{A}_E)} \int_{Z'(\mathbb{A}_F)H''(F)\backslash H''(\mathbb{A}_F)} K_{f'}(h, g)\, \chi_{H'}(h)(\chi\eta)^{-1}(g)\mathrm{d}h\mathrm{d}g.$$

The motivation for introducing this distribution will be clear when we discuss its spectral expansion in Subsection 3.2.

We say that an element $x \in G'(F)$ is regular semisimple if the stabilizer $(H' \times H'')_x$ is a torus of dimension n over F. It is elliptic if in addition $(H' \times H'')_x$ is an elliptic torus. Let $G'(F)_{\mathrm{ell}}$ be the subset of elliptic elements. These definitions also apply to elements in $G'(F_v)$ where v is a place of F.

Assume that $f' = \otimes f'_v$ is decomposable and there is one nonsplit place v_1 of F such that f'_{v_1} is supported in the regular elliptic locus. We fix a character $\widetilde{\eta} : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\widetilde{\eta}|_{\mathbb{A}_F^\times} = \eta$. Then as usual the distribution I unfolds to orbital integrals, i.e. we have

$$I(f') = \sum_{x \in H'(F)\backslash G'(F)_{\mathrm{ell}}/H''(F)} \operatorname{vol}((H' \times H'')_x) O^{G'}(x, f'), \tag{2.2}$$

where

$$O^{G'}(x, f') = \int_{(H' \times H'')_x(\mathbb{A}_F)\backslash(H' \times H'')(\mathbb{A}_F)} f'(h^{-1}xh'')(\chi_{H'}\chi^{-1}\widetilde{\eta}^{-1})(h)(\chi\widetilde{\eta})^{-1}(h^{-1}xh'')\mathrm{d}h\mathrm{d}h''.$$

We fix compatible measures on $Z'(\mathbb{A}_F)\backslash(H' \times H'')_x(\mathbb{A}_F)$, $Z'(\mathbb{A}_F)\backslash(H' \times H'')(\mathbb{A}_F)$ and $(H' \times H'')_x(\mathbb{A}_F)\backslash(H' \times H'')(\mathbb{A}_F)$ for each $x \in G'(F)_{\mathrm{ell}}$ and $\operatorname{vol}((H' \times H'')_x)$ stands for the volume of

$$Z'(\mathbb{A}_F)(H' \times H'')_x(F)\backslash(H' \times H'')(\mathbb{A}_F).$$

This integral is absolutely convergent for all regular semisimple x. Since the test function f' is not compactly supported, the absolute convergence of (2.2) needs explanation. This will be given in the Appendix A.

If v is a place of F, then we define similarly the local orbital integral, except we integrate over $(H' \times H'')(F_v)$ instead.

The orbital integral can be simplified as follows. Introduce the symmetric space

$$S' = \{s\bar{s} = 1 \mid s \in G'\} \simeq G'/H'',$$

on which H' acts by twisted conjugation. Put

$$(2.3) \qquad \widetilde{f}'(g\overline{g}^{-1}) = \int_{H''(\mathbb{A}_F)} f'(gh)(\chi\widetilde{\eta})^{-1}(gh)\mathrm{d}h.$$

Then $\widetilde{f}' \in \mathcal{S}(S'(\mathbb{A}_F))$. We have

$$(2.4) \qquad O^{G'}(g, f') = O^{S'}(s', \widetilde{f}') = \int_{H'_{s'}(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)} \widetilde{f}'(h^{-1}s'\overline{h})(\chi_{H'}\chi^{-1}\widetilde{\eta}^{-1})(h)\mathrm{d}h, \quad s' = g\overline{g}^{-1}.$$

If v is a place of F, the local orbital integral can be defined and simplified in a similar way.

An element $s' \in S'(F)$ is regular semisimple if $H'_{s'}$ is a torus of dimension n. It is in addition elliptic if $H'_{s'}$ is an anisotropic torus modulo the split center of G'. If $s' = g\overline{g}^{-1}$, $g \in G'(F)$, then s' is regular semisimple or elliptic if g is so in $G'(F)$. Let $S'(F)_{\mathrm{reg}}$ and $S'(F)_{\mathrm{ell}}$ be the subsets of regular semisimple and elliptic elements respectively.

2.2. **Matching of test functions.** Recall that we fixed $\epsilon \in NE^\times$ (resp. $F^\times\backslash NE^\times$) if D splits (resp. ramifies) and the group G is realized as a subgroup of $\mathrm{GL}_{2n}(E)$ consisting of matrices of the form

$$\begin{pmatrix} \alpha & \epsilon\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha, \beta \in M_n(E).$$

Then H consists of matrices of the form $\begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix}$, $\alpha \in \mathrm{GL}_n(E)$.

Let $x \in G'(F)$ and $y \in G(F)$ be regular semisimple elements. Write

$$x\overline{x}^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in S'(F), \quad y\theta(y)^{-1} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \in S(F),$$

where $\alpha_i, \beta_i \in M_n(E)$. We will prove that α_i's and β_i's are all invertible. We say that x and y match if $2\alpha_1\overline{\alpha_1} - 1$ and β_1 have the same characteristic polynomial. The general form of such matching comes from the consideration of categorical quotients, cf. Section 5, and the exact form of the matching is only obtained during the attempt to prove the fundamental lemma, cf. Section 7. Matching of regular semisimple elements will be studied in detail in Section 6, and it turns out that not all regular semisimple $x \in G'(F)$ matches a regular semisimple $y \in G(F)$, and vice versa. This is a new feature of the present relative trace formula at hand. The element x matches some y (resp. y matches some x) if

$$1 - (\alpha_1\overline{\alpha_1})^{-1} \in \epsilon N\,\mathrm{GL}_n(E), \quad \text{resp.} \quad \frac{1}{2}(\beta_1 + 1) \in N\,\mathrm{GL}_n(E).$$

The matching of regular semisimple elements also applies to the situation of F_v where v is a place of F. We note that there is a neighbourhood of $1 \in G(F_v)$ such that every regular semisimple y in this neighbourhood matches some $x \in G'(F_v)$. This is because there is a small neighbourhood of $1 \in \mathrm{GL}_n(E_v)$ such that every element in this neighbourhood is a norm.

If $x \in G'(F_v)$ and $y \in G(F_v)$ match, we will see in Subsection 6.1 that the stabilizers $(H' \times H'')_x(F_v)$ and $(H \times H)_y(F_v)$ are isomorphic and we fix such an isomorphism. We fix measures on

9

these stabilizers which are identified under the fixed isomorphism. If $x \in G'(F)$ and $y \in G(F)$ match, then they match at every place of F. With the above fixed measures on their stabilizers we have

$$\mathrm{vol}((H' \times H'')_x) = \mathrm{vol}((H \times H)_y).$$

Fix an element $\tau \in E^\times$ such that $\bar{\tau} = -\tau$. For each place v of F, we define transfer factors on G' and G:

(2.5)
$$\kappa_v^{G'}(x) = \chi_v(\alpha_4)\widetilde{\eta}_v(\tau\alpha_2), \quad \kappa_v^G(y) = \chi_v(y_1), \quad y^{-1} = \begin{pmatrix} y_1 & \epsilon y_2 \\ \overline{y_2} & \overline{y_1} \end{pmatrix}.$$

For each place v of F, we define

$$\mathcal{S}(G'(F_v))_0 = \{f' \in \mathcal{S}(G'(F_v)) \mid O^{G'}(x, f'_v) = 0 \text{ for all } x \text{ not matching any } y \in G(F_v)\},$$

and $\mathcal{S}(G(F_v))_0$ in a similar way. By definition, two test functions $f'_v \in \mathcal{S}(G'(F_v))_0$ and $f_v \in \mathcal{S}(G(F_v))_0$ match if and only if

$$\kappa_v^{G'}(x)O^{G'}(x, f'_v) = \kappa_v^G(y)O^G(y, f_v)$$

for all matching regular semisimple $x \in G'(F_v)$ and $y \in G(F_v)$. Two test functions $f' = \otimes f'_v \in \mathcal{S}(G'(\mathbb{A}_F))$ and $f = \otimes f_v \in \mathcal{S}(G(\mathbb{A}_F))$ match if $f'_v \in \mathcal{S}(G'(F_v))_0$ and $f_v \in \mathcal{S}(G(F_v))_0$ and they match for all place v of F.

The following are the main theorems concerning the geometric side of the trace formulae. They will be proved in Sections 6 and 7 respectively.

Theorem 2.1. *Assume that v is nonarchimedean and nonsplit. For any $f'_v \in \mathcal{S}(G'(F_v))_0$ there is an $f_v \in \mathcal{S}(G(F_v))_0$ that matches it, and vice versa.*

Theorem 2.2. *Let v be a nonsplit nonarchimedean odd place in F. Assume the quaternion algebra D splits at v and χ_v is unramified at v. Let $\mathfrak{o}_{F_v}$ be the ring of integers of F_v. We pick the measure on $G'(F_v)$ and $G(F_v)$ such that the volumes of $G'(\mathfrak{o}_v)$ and $G(\mathfrak{o}_v)$ are 1. Then $\mathbf{1}_{G'(\mathfrak{o}_{F_v})}$ and $\mathbf{1}_{G(\mathfrak{o}_{F_v})}$ match.*

2.3. **Matching at split places.** The goal of this subsection is to explain the matching of test functions at the split places, which can be made explicit.

If v is a split place of F. Then $G(F_v) = \mathrm{GL}_{2n}(F_v)$ and $G'(F_v) = \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_v)$. We fix a measure on $\mathrm{GL}_{2n}(F_v)$ and thus we have measures on $G'(F_v)$ and on $G(F_v)$ under this identification. The character η_v is trivial so $\widetilde{\eta}_v$ takes the form (η_0, η_0^{-1}) where η_0 is a character of $F_v^\times$. The character χ_v is of the form (χ_1, χ_2) where χ_1, χ_2 are characters of $F_v^\times$. Regular semisimple elements $y \in G(F_v)$ and $(x_1, x_2) \in G'(F_v)$ matches if $y = x_1 x_2^{-1}$. Let $f' = (f'_1, f'_2) \in \mathcal{S}(G'(F_v)) \simeq \mathcal{S}(G(F_v)) \otimes \mathcal{S}(G(F_v))$ and put

(2.6)
$$f(g) = \int_{\mathrm{GL}_{2n}(F_v)} f'_1(gh)f'_2(h)\chi_1^{-1}(h)\chi_2^{-1}(h)\mathrm{d}g, \quad g \in \mathrm{GL}_{2n}(F_v).$$

10

Lemma 2.3. *The functions f' and f match.*

Proof. We write

$$x_1 x_2^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad x_2 x_1^{-1} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

A little computation gives that the orbital integral $\kappa^{G'}((x_1, x_2)) O^{G'(F_v)}((x_1, x_2), f')$ equals

$$\eta_0(x_1 x_2^{-1})^{-1} \eta_0(-B_1 B_2^{-1}) \chi_1(D_1) \chi_2(D_2) \chi_1(x_1 x_2^{-1})^{-1}$$

(2.7)
$$\int f\left(\begin{pmatrix} a_1 & \\ & b_1 \end{pmatrix}^{-1} x_1 x_2^{-1} \begin{pmatrix} a_2 & \\ & b_2 \end{pmatrix} \right) \chi_1(a_1 a_2^{-1}) \chi_2(b_1 b_2^{-1}) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}b_1 \mathrm{d}b_2.$$

When (x_1, x_2) is regular semisimple, A_2 is invertible, and thus

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} 1 & \\ C_2 A_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ & D_2 - C_2 A_2^{-1} B_2 \end{pmatrix}.$$

Thus $D_1 = (D_2 - C_2 A_2^{-1} B_2)^{-1}$ and $(\det x_1 x_2^{-1})^{-1} = \det A_2 \det(D_2 - C_2 A_2^{-1} B_2)$. It follows that $\chi_1(D_1)\chi_2(D_2)\chi_1(x_1 x_2^{-1})^{-1} = \chi_1(A_2)\chi_2(D_2)$. Similarly we have $\eta_0(x_1 x_2^{-1}) = \eta_0(-B_1 B_2^{-1})$. In particular we note that $\kappa^{G'}((x_1,x_2)) O^{G'(F_v)}((x_1,x_2), f')$ is independent of the choice of $\widetilde{\eta}$. A straightforward computation then gives that (2.7) equals

$$\kappa^G(y) O^G(y, f),$$

when $y = x_1 x_2^{-1}$. $\qquad \square$

When we speak of the matching of test functions f' and f at split places, we always mean that f' and f are related in this explicit way.

We observe that there is an involution on $\mathcal{S}(G(F_v))$ given by $f \mapsto f^\vee \chi_1 \chi_2$. Define

$$\mathcal{S}(G(F_v))^+ = \{f \in \mathcal{S}(G(F_v)) \mid f^\vee \chi_1 \chi_2 = f\}, \quad \mathcal{S}(G'(F_v))^+ = \mathcal{S}(G(F_v))^+ \otimes \mathcal{S}(G(F_v))^+.$$

There is a base change homomorphism

$$\mathrm{bc}_v : \mathcal{S}(G'(F_v)) \simeq \mathcal{S}(G(F_v)) \otimes \mathcal{S}(G(F_v)) \to \mathcal{S}(G(F_v)),$$

which is given by the usual multiplication (i.e. convolution) on $\mathcal{S}(G(F_v))$. It maps $\mathcal{S}(G'(F_v))^+$ to $\mathcal{S}(G(F_v))^+$, cf. [AC89, Chapter 1, Section 5].

Lemma 2.4. *If $f' \in \mathcal{S}(G'(F_v))^+$ then f' and $\mathrm{bc}_v(f')$ match.*

Proof. This is a direct consequence of Lemma 2.3 and the definition of $\mathcal{S}(G'(F_v))^+$. $\qquad \square$

Let us assume further that v is an odd unramified (split) finite place, and χ_1, χ_2 are unramified characters of $F_v^\times$. Let $K_v = \mathrm{GL}_{2n}(\mathfrak{o}_{F_v})$ and $\mathcal{H}_v = \mathbb{C}[K_v \backslash G(F_v)/K_v]$ be the spherical Hecke algebra of $G(F_v)$. Then the involution $f \mapsto f^\vee \chi_1 \chi_2$ reduces to an involution on $\mathcal{H}_v$. Let $\mathcal{H}_v^\pm$ be the ± 1-eigenspaces of this involution. Similarly let $\mathcal{H}_{E,v}$ be the spherical Hecke algebra of $G'(F_v)$, which is identified with $\mathcal{H}_v \otimes \mathcal{H}_v$.

11

Lemma 2.5. *If*

$$f'_v = (f'_1, f'_2) \in (\mathcal{H}_v^- \otimes \mathcal{H}_v^+) \oplus (\mathcal{H}_v^+ \otimes \mathcal{H}_v^-) \oplus (\mathcal{H}_v^- \otimes \mathcal{H}_v^-),$$

then we have

$$O^{G'(F_v)}((x_1, x_2), f') = 0, \quad O^{G(F_v)}(y, \mathrm{bc}_v(f')) = 0.$$

Proof. From the explicit expressions (2.7), it suffices to check that if $f \in \mathcal{H}_v^-$, then

$$(2.8) \qquad \int_{\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)} f\left(\begin{pmatrix} a & \\ & b \end{pmatrix} g\right) \chi_1(a)^{-1} \chi_2(b)^{-1} \mathrm{d}a \mathrm{d}b = 0$$

for all $g \in G(F_v)$. By [Off11, Proposition 3.1], every double coset $H(F_v) \backslash G(F_v) / K_v$ is represented by an element g such that

$$g^{-1}\theta(g) = \begin{pmatrix} & \Lambda \\ -\Lambda^{-1} & \end{pmatrix}, \quad \Lambda = \begin{pmatrix} & & \varpi^{\lambda_1} \\ & \cdot^{\cdot^{\cdot}} & \\ \varpi^{\lambda_n} & & \end{pmatrix}, \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

where ϖ is a uniformizer in F_v. Therefore to see (2.8) it is enough to assume that g is of this form. We may further assume that g take the shape $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and A, D are diagonal while B, C consist of only anti-diagonal entries, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, where a, b, c, d are entries of g at (i, i), $(i, n - i + 1)$, $(n - i + 1, i)$ and $(n - i + 1, n - i + 1)$ respectively. With this g it is straightforward to see that

$$(2.9) \qquad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} {}^t g^{-1} = g \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Note that since $f \in \mathcal{H}_v$ we have $f({}^t g) = f(g)$ for all g. Now replacing f by $-f^\vee \chi_1 \chi_2$ in (2.8), we see that

$$(2.8) = -\int f\left({}^t\begin{pmatrix} a & \\ & b \end{pmatrix}^{-1} {}^t g^{-1}\right) \chi_2(a) \chi_1(b) \mathrm{d}a \mathrm{d}b$$

$$= -\int f\left({}^t\begin{pmatrix} a & \\ & b \end{pmatrix}^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) \chi_2(a) \chi_1(b) \mathrm{d}a \mathrm{d}b.$$

Make change of variables $a \mapsto {}^t b^{-1}$ and $b \mapsto {}^t a^{-1}$, and use the fact that $f \in \mathcal{H}_v$ we see that the last integral equals $-(2.8)$, which implies (2.8) equals zero. $\qquad \square$

Proposition 2.6. *The function $f' \in \mathcal{H}_{E,v}$ and $f = \mathrm{bc}_v(f') \in \mathcal{H}_v$ match.*

Proof. By Lemma 2.5 above, the assertion holds if $f'_v = (f'_1, f'_2) \in (\mathcal{H}_v^- \otimes \mathcal{H}_v^+) \oplus (\mathcal{H}_v^+ \otimes \mathcal{H}_v^-) \oplus (\mathcal{H}_v^- \otimes \mathcal{H}_v^-)$. If $f'_v \in \mathcal{H}_v^+ \otimes \mathcal{H}_v^+$ then the map bc_v and the map (2.6) coincide. $\qquad \square$

3. Relative trace formulae: the spectral side

3.1. **Linear periods and Shalika models.** We recall some results on (split) linear periods and Shalika models on $\mathrm{GL}_{2n}(\mathbb{A}_E)$ from [FJ93, JS90]. In this subsection we temporarily use the following notation. Let Q be the Shalika subgroup of $\mathrm{GL}_{2n}(E)$ consisting of matrices of the form

$$\begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad g \in \mathrm{GL}_n(E), \quad x \in M_n(E).$$

Let $\psi : E \backslash \mathbb{A}_E \to \mathbb{C}^\times$ be a nontrivial additive character and $\chi, \xi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ be two characters. Define a character $\theta : Q(E) \backslash Q(\mathbb{A}_E) \to \mathbb{C}^\times$ by

$$\theta \left(\begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) = \psi(\mathrm{Tr}\, x) \xi(\det g).$$

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ with central character ω, and assume that $\xi^n \omega = 1$.

We define global Shalika functional on Π

$$\lambda(\varphi) = \int_{\mathbb{A}_E^\times Q(E) \backslash Q(\mathbb{A}_E)} \varphi(s) \theta(s) \mathrm{d}s, \quad \varphi \in \Pi.$$

By [JS90, Section 8, Theorem 1], the Shalika functional on Π is not identically zero if and only if $L(s, \Pi, \wedge^2 \otimes \xi)$ has a simple pole at $s = 1$. If this is the case, for $\varphi \in \Pi$ we put

$$V_\varphi(g) = \lambda(\Pi(g)\varphi), \quad g \in \mathrm{GL}_{2n}(\mathbb{A}_E).$$

Following [FJ93] we consider the integral

$$Z^{\mathrm{FJ}}(s, \varphi, \chi, \xi) = \int \varphi \left(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \right) |\det h_1 h_2^{-1}|^{s - \frac{1}{2}} \chi(\det h_1 h_2^{-1}) \xi(\det h_2) \mathrm{d}h_1 \mathrm{d}h_2,$$

Here the integration is over

$$(h_1, h_2) \in \mathbb{A}_E^\times (\mathrm{GL}_n(E) \times \mathrm{GL}_n(E)) \backslash (\mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_E))$$

and $\mathbb{A}_E^\times$ stands for the center of $\mathrm{GL}_{2n}(\mathbb{A}_E)$. The integral is convergent for all $s \in \mathbb{C}$. By [FJ93, Proposition 2.2] this integral is not identically zero only if the Shalika function on Π is not identically zero. Assume that this is the case. Then by [FJ93, Proposition 2.3] when $\mathrm{Re}\, s$ is sufficiently large this integral unfolds to

$$\int_{\mathrm{GL}_n(\mathbb{A}_E)} V_\varphi \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \chi(\det g) |\det g|^{s - \frac{1}{2}} \mathrm{d}g.$$

By [FJ93, Theorem 4.1] this integral equals a holomorphic multiple of $L(s, \Pi \otimes \chi)$, and there is a choice of φ such that it equals this L-function. Specializing to the point $s = \frac{1}{2}$, we obtain the following proposition.

Proposition 3.1. *We have $Z^{\mathrm{FJ}}(\frac{1}{2}, \varphi, \chi, \xi) \neq 0$ for some $\varphi \in \Pi$ if and only if $L(s, \Pi, \wedge^2 \otimes \xi)$ has a simple pole at $s = 1$ and $L(\frac{1}{2}, \Pi \otimes \chi) \neq 0$.*

Let us now switch to the local situation. Let v be a place of E. Define the character $\theta_v : Q(E_v) \to \mathbb{C}^\times$ the same way as the global case. By [CS20], the space

$$\mathrm{Hom}_{Q(E_v)}(\Pi_v \otimes \theta_v, \mathbb{C})$$

is at most one dimensional. Assume that it is one dimensional and λ_v is a nonzero element. For any $\varphi_v \in \Pi_v$ put

$$V_{\varphi_v}(g) = \lambda(\Pi_v(g)\varphi_v), \quad g \in \mathrm{GL}_{2n}(E_v),$$

and

$$\mathcal{V}_v = \{V_{\varphi_v} \mid \varphi_v \in \Pi_v\}.$$

The space $\mathcal{V}_v$ is called the local Shalika model of Π_v. As in [FJ93, Section 3] we define

$$Z_v^{\mathrm{FJ}}(s, V_v, \chi_v) = \int_{\mathrm{GL}_n(F_v)} V_v\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) \chi_v(g)|\det g|^{s-\frac{1}{2}} \mathrm{d}g.$$

Here we note that this implicitly depends on ψ_v and ξ_v through the Shalika model. Then by [FJ93, Proposition 3.1], it is a holomorphic multiple of $L(s, \Pi_v \otimes \chi_v)$ and there is a $V_v \in \mathcal{V}_v$ such that it equals this local L-function. Moreover by [FJ93, Proposition 3.3] there is a functional equation

$$\gamma(s, \Pi_v \otimes \chi_v, \psi_v)Z_v^{\mathrm{FJ}}(s, V_v, \chi_v) = Z_v^{\mathrm{FJ}}(1-s, \widetilde{V_v}, \chi_v^{-1}),$$

where

$$\widetilde{V_v}(g) = V_v\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} {}^t g^{-1}\right),$$

and $\widetilde{V_v}$ belong to the Shalika model of the contragredient of π (defined using the characters χ_v^{-1} and ψ_v^{-1}). Specializing to $s = \frac{1}{2}$ we conclude that

$$(3.1) \qquad \epsilon(\Pi_v \otimes \chi_v, \psi_v)\chi_v(-1)^n Z_v^{\mathrm{FJ}}(\frac{1}{2}, V_v, \chi_v) = Z_v^{\mathrm{FJ}}(\frac{1}{2}, \widetilde{V_v}, \chi_v^{-1}),$$

where $\epsilon(\Pi_v \otimes \chi_v, \psi_v)$ is the local root number of $\Pi_v \otimes \chi_v$. Note that this is not exactly the same as [FJ93, Proposition 3.3] (there is an extra factor $\chi_v(-1)^n$). This is due to the fact that the function $\widetilde{V_v}$ in [FJ93, Proposition 3.3] is defined by left multiplication by $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ in our definition it is $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

3.2. **Functoriality.** We return to the setup and notation of Section 2. Let Π be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$. We define $\Pi^\vee$ to be the dual of Π, and Π^c to be the Galois conjugate (relative to E/F) of Π, i.e. the automorphic representation whose space is given by $\{\varphi(\overline{g}) \mid \varphi \in \Pi\}$. Then it is well-known that $\Pi^c \simeq \Pi$ if and only if $\Pi = \pi_E$ for some cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, cf. [AC89, Chapter 3, Theorem 4.2 and 5.1].

We apply the results recalled the previous subsection to the current situation. Put

$$(3.2) \qquad P'_\chi(\varphi) = \int_{Z'(\mathbb{A}_F)H'(F)\backslash H'(\mathbb{A}_F)} \varphi\left(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}\right) \chi(h_1 \overline{h_2}) \mathrm{d}h_1 \mathrm{d}h_2$$

Then by Proposition 3.1, applied to the case $\xi = \chi\chi^c$, the linear form P'_χ is not identically zero if and only if $L(\frac{1}{2}, \Pi \otimes \chi) \neq 0$ and $L(s, \Pi, \wedge^2 \otimes \chi\chi^c)$ has a pole at $s = 1$. The later implies that $\Pi^\vee \simeq \Pi \otimes \chi\chi^c$.

We also consider the Flicker–Rallis period of Π given by

$$P''_{\chi\eta}(\varphi) = \int_{Z_{2n}(\mathbb{A}_F)H''(F)\backslash H''(\mathbb{A}_F)} \varphi(h)(\chi\eta)(h)\mathrm{d}h.$$

It is not identically zero if and only if the Asai L-function $L(s, \Pi \otimes \chi, \mathrm{As}^-)$ has a pole at $s = 1$, cf. [Kab04]. Note that this implies that $\Pi^\vee \otimes \chi^{-1} \simeq \Pi^c \otimes \chi^c$.

Lemma 3.2. *If neither of P'_χ and $P''_{\chi\eta}$ is identically zero, then there is an irreducible cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ such that $\Pi = \pi_E$. Moreover, $L(\frac{1}{2}, \Pi \otimes \chi) \neq 0$, and $L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times})$ has a simple pole at $s = 1$.*

Proof. The existence of π follows from the above discussion. To prove the second assertion, let us observe that

$$L(s, \Pi \otimes \chi, \mathrm{As}^-) = L(s, \pi, \mathrm{Sym}^2 \otimes\chi|_{\mathbb{A}_F^\times}\eta)L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times}).$$

Indeed this can be checked place by place for almost all places of F. If v is split in E, then the equality clearly holds. Assume v is inert and Π_v (and hence π_v) is unramified. Let q_v be the cardinality of the residue field of F at v. Let $\beta_1, \cdots, \beta_{2n}$ be the Satake parameters of π_v, and $\gamma = \chi_v(\varpi_v)$ where ϖ_v is the uniformizer at v. Then the Satake parameters of Π_v are $\beta_1^2, \cdots, \beta_{2n}^2$. Thus the left hand side equals

$$\prod_{1 \leq i \leq 2n} (1 + \beta_i^2 \gamma q_v^{-s})^{-1} \prod_{1 \leq i < j \leq 2n} (1 - \beta_i^2 \beta_j^2 \gamma^2 q_v^{-2s})^{-1}.$$

The right hand side equals

$$\prod_{1 \leq i \leq j \leq 2n} (1 + \beta_i \beta_j \gamma q_v^{-s})^{-1} \prod_{1 \leq i < j \leq 2n} (1 - \beta_i \beta_j \gamma q_v^{-s})^{-1}.$$

Thus the desired equality holds at the place v.

Similar we also have

$$L(s, \Pi, \wedge^2 \otimes \chi\chi^c) = L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times})L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times}\eta).$$

If $L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times})$ is holomorphic at $s = 1$, then both

$$L(s, \pi, \mathrm{Sym}^2 \otimes\chi|_{\mathbb{A}_F^\times}\eta), \quad L(s, \pi, \wedge^2 \otimes \chi|_{\mathbb{A}_F^\times}\eta)$$

have a pole at $s = 1$, which implies that

$$L(s, \pi \times \pi\chi|_{\mathbb{A}_F^\times}\eta)$$

has at least a double pole at $s = 1$, which is not possible. This proves the final assertion. $\qquad \square$

3.3. **Local spherical charaters.** Let v be a nonarchimedean nonsplit place. Let π_v be an irreducible unitary representation of $G(F_v)$. According to [Lu], the space $\mathrm{Hom}_{H(F_v)}(\pi_v \otimes \chi_v, \mathbb{C})$ is at most one dimensional. We assume that it is one dimensional and we fix a nonzero element ℓ in it. Let $f \in \mathcal{S}(G(F_v))$ and we consider the spherical character

$$J_{\pi_v}(f) = \sum_\phi \ell(\pi_v(f)\phi)\overline{\ell(\phi)}$$

where the sum runs over an orthonormal basis of π. A test functions of the form $f = f_1 * \overline{f_1^\vee}$ is called of positive type.

Lemma 3.3. *There is a positive type $f \in \mathcal{S}(G(F_v))_0$ such that f is supported in a sufficiently small neighbourhood of $1 \in G(F_v)$ and $J_{\pi_v}(f) > 0$.*

Proof. Choose $\phi \in \pi_v$ such that $\ell(\phi) \neq 0$. Let $f_1 \in \mathcal{S}(G(F_v))$ be the characteristic function of a small open compact subgroup, then $\ell(\pi(f_1)\phi) \neq 0$. Put $f = f_1 * \overline{f_1^\vee}$. Then f is supported in a sufficiently small neighbourhood of $1 \in G(F_v)$ and thus $f \in \mathcal{S}(G(F_v))_0$, because every regular semisimple element in the support of f matches some element in $G'(F_v)$. Moreover we have

$$J_{\pi_v}(f) = \sum_\phi \ell(\pi_v(f_1)\phi)\overline{\ell(\pi_v(f_1)\phi)}.$$

Each term in the sum is nonnegative and there is at least one nonzero term. Therefore $J_{\pi_v}(f) > 0$. $\qquad \square$

By [Guo98], J_{π_v} is represented by a locally integrable function Θ_{π_v} on $G(F_v)$, which is locally constant on the regular semisimple locus, and which satisfies the invariance property $\Theta_{\pi_v}(h_1 g h_2) = \chi_v(h_1 h_2)\Theta_\pi(g)$. We say that π_v is $H(F_v)$-elliptic $\Theta_{\pi_v}(y) \neq 0$ for some elliptic regular semisimple $y \in G(F_v)$ and y matches some $x \in G'(F_v)$.

Proposition 3.4. *If π_v is supercuspidal and $\mathrm{Hom}_{H(F_v)}(\pi_v \otimes \chi_v, \mathbb{C}) \neq 0$, then π_v is $H(F_v)$-elliptic.*

The argument is standard and very close to the classical theory of Harish-Chandra. The main part of the proof has also been worked out in a slightly different setting in [Xue22]. A detailed proof is quite long, and it produces a lot of repetitions from the literature and deviates significantly from the goal of this paper. So we will merely sketch the argument in Appendix B.

3.4. **Involution.** We need to consider the local spherical characters arising from the distribution I_Π. Let us fix a nonsplit nonarchimedean place v. Then we have

$$\mathrm{Hom}_{H'(F_v)}(\Pi_v \otimes \chi_{H',v}, \mathbb{C}) \neq 0, \quad \mathrm{Hom}_{H''(F_v)}(\Pi_v \otimes \chi_v \eta_v, \mathbb{C}) \neq 0.$$

We let ℓ' and ℓ'' be nonzero elements in these Hom-spaces respectively. Define the local spherical character

$$I_{\Pi_v}(f') = \sum_W \ell'(\Pi_v(f')W)\overline{\ell''(W)}, \quad f' \in \mathcal{S}(G'(F_v)),$$

where W ranges over an orthonormal basis of Π_v.

We say that I_{Π_v} is elliptic if there is an f' supported in the elliptic locus of $G'(F_v)$ such that $I_{\Pi_v}(f') \neq 0$. We expect that all supercuspidal representations are elliptic. We will address it in a subsequent paper.

Let $f' \in \mathcal{S}(G'(F_v))$. Put

(3.3) $$f'^\dagger(g) = f'({}^t\overline{g}^{-1})(\chi_v\chi_v^c)(g), \quad g \in G'(F_v).$$

Let ψ_v be a nontrivial additive character of E_v trivial on F_v. Since $\Pi_v^\vee \otimes \chi_v^{-1} \simeq \Pi_v^c \otimes \chi_v^c$, the local root number $\epsilon(\Pi_v \otimes \chi_v, \psi_v) = \pm 1$ and is independent of ψ_v. We denote it by $\epsilon(\Pi_v \otimes \chi_v)$.

Lemma 3.5. *For any $f' \in \mathcal{S}(G'(F_v))$ we have*

$$I_{\Pi_v}(f'^\dagger) = \chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v)I_{\Pi_v}(f').$$

Proof. To prove this lemma, we need to choose ℓ' and ℓ'' more carefully. Since different choices differ only by a constant, the validity of the lemma is independent of such choices.

Let $\mathcal{W}$ be the Whittaker model of Π_v (defined by the character ψ_v) and $\mathcal{V}$ be the Shalika model of Π_v defined by the characters ψ_v and $\chi_v\chi_v^c$. We fix an isomorphism

$$\mathcal{W} \to \mathcal{V}, \quad W \mapsto \phi_W.$$

We also denote by $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{V}}$ the Whittaker model and Shalika model of $\pi^\vee$, defined using the characters ψ_v^{-1} and ψ_v^{-1}, $(\chi_v\chi_v^c)^{-1}$ respectively. We also fix an isomorphism

$$\widetilde{\mathcal{W}} \to \widetilde{\mathcal{V}}, \quad W \mapsto \phi_W.$$

For any function α on $G'(F_v)$ we temporarily put

$$\widetilde{\alpha}(g) = \alpha\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} {}^t g^{-1}\right).$$

Then both

$$W \mapsto \phi_W, \quad W \mapsto \widetilde{\phi_{\widetilde{W}}}$$

are isomorphisms between $\mathcal{W}$ and $\mathcal{V}$, and hence they differ by a constant. By rescaling the isomorphisms that we have fixed, we may assume that this constant equals one. It follows that for any $W \in \mathcal{W}$, we have

$$\phi_{\widetilde{W}} = \widetilde{\phi_W}.$$

For any $W \in \mathcal{W}$ we put $W^c(g) = W(\overline{g})$.

Recall from Subsection 3.1 that we have the integral

$$Z^{\text{FJ}}(s, \phi, \chi_v) = \int_{\text{GL}_n(E_v)} \phi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) \chi_v(a) |\det a|^{s - \frac{1}{2}} \mathrm{d}a, \quad \phi \in \mathcal{V}.$$

We choose $\ell' \in \text{Hom}_{H'(F_v)}(\Pi_v \otimes \chi_v, \mathbb{C})$ to be the

$$W \mapsto \ell'(W) = Z^{\text{FJ}}(\frac{1}{2}, \phi_W, \chi_v).$$

For the linear form ℓ'', we take it to be

$$\ell''(W) = \int_{N' \cap H''(F_v) \backslash P' \cap H''(F_v)} W(h)(\chi_v \eta_v)(h) \mathrm{d}h.$$

where P' is the mirabolic subgroup of G' and N' the standard upper triangular unipotent subgroup of G'.

If $W \in \mathcal{W}$, since $\Pi_v^\vee \simeq \Pi_v \otimes (\chi_v \chi_v^c)^{-1}$, we have $\widetilde{W^c}(\chi_v \chi_v^c)^{-1} \in \mathcal{W}$. We claim that for any $g \in G'(F_v)$ we have

(3.4) $$\chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v) \ell'(\Pi_v({}^t\overline{g}^{-1})W)(\chi_v \chi_v^c)^{-1}(g) = \ell'(\Pi_v(g)(\widetilde{W^c}(\chi_v \chi_v^c)^{-1})).$$

The right hand side equals

$$\int \phi_{\Pi_v(g)(\widetilde{W^c}(\chi_v \chi_v^c)^{-1})}\left(\begin{pmatrix} a & \\ & 1_n \end{pmatrix}\right) \chi_v(a) \mathrm{d}a$$

which simplifies to

$$(\chi_v \chi_v^c)^{-1}(g) \int \phi_{\Pi_v({}^t\overline{g}^{-1})W}\left(\begin{pmatrix} a & \\ & 1_n \end{pmatrix}\right) \chi_v^{-1}(a) \mathrm{d}a$$

Using the functional equation (3.1) of Z^{FJ} we conclude that this equals

$$\chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v)(\chi_v \chi_v^c)^{-1}(g) \int \phi_{\Pi_v({}^t\overline{g}^{-1})W}\left(\begin{pmatrix} a & \\ & 1_n \end{pmatrix}\right) \chi_v(a) \mathrm{d}a,$$

which is precisely the left hand side of (3.4).

We now compute $I_{\Pi_v}(f'^\dagger)$. By definition we have

$$\ell'(\Pi_v(f'^\dagger)W) = \int_{G'(F)} f'({}^t\overline{g}^{-1}) \ell'(\Pi_v(g)W)(\chi_v \chi_v^c)(g) \mathrm{d}g$$

$$= \int_{G'(F)} f'(g) \ell'(\Pi_v({}^t\overline{g}^{-1})W)(\chi_v \chi_v^c)(g)^{-1} \mathrm{d}g$$

$$= \chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v) \int_{G'(F)} f'(g) \ell'(\Pi_v(g)(\widetilde{W^c}(\chi_v \chi_v^c)^{-1})).$$

Thus we have

$$\ell'(\Pi_v(f'^\dagger)W) = \chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v) \ell'(\Pi_v(f')(\widetilde{W^c}(\chi_v \chi_v^c)^{-1})).$$

By [LM14, Lemma 1.1] (the main part of the proof is from [Off11, Corollary 7.2]) we have

$$\ell''(\widetilde{W^c}(\chi_v \chi_v^c)^{-1}) = \ell''(W).$$

18

Thus we conclude

$$I_{\Pi_v}(f'^\dagger) = \chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v) \sum_W \ell'(\Pi_v(f')(\widetilde{W}^c(\chi_v\chi_v^c)^{-1}))\overline{\ell''(\widetilde{W}^c(\chi_v\chi_v^c)^{-1})}.$$

When W ranges over an orthonormal basis in $\mathcal{W}$, $\widetilde{W}^c(\chi_v\chi_v^c)^{-1}$ ranges over an orthonormal basis in $\mathcal{W}$ as well. Thus

$$I_{\Pi_v}(f'^\dagger) = \chi_v(-1)^n \epsilon(\Pi_v \otimes \chi_v) I_{\Pi_v}(f').$$

This proves the lemma. □

We now study the relation between matching of test functions and this involution. Let $\epsilon_{D_v} = \eta_v(\epsilon) = \pm 1$.

Lemma 3.6. *Let $f' \in \mathcal{S}(G'(F_v))_0$ and $f \in \mathcal{S}(G(F_v))_0$ be matching test functions. Then $f'^\dagger$ and $\eta_v(-1)^n \epsilon_{D_v}^n f$ match.*

Proof. Let $x \in G'(F)$ and $s' = x\overline{x}^{-1}$. Then $O^{G'}(x, f') = O^{S'}(s', \widetilde{f}')$ where $\widetilde{f}'$ is defined by (2.3). Moreover a little computation shows that

$$O^{G'}(x, f'^\dagger) = O^{S'}({}^t s', \widetilde{f}').$$

We will see below in Section 5.1 that s' is in the H'-orbit of the form

$$\begin{pmatrix} \alpha & 1 \\ 1 - \alpha\overline{\alpha} & -\overline{\alpha} \end{pmatrix}.$$

So we may assume that s' equals it. Then

$${}^t s' = \begin{pmatrix} \alpha & 1 - \alpha\overline{\alpha} \\ 1 & -\overline{\alpha} \end{pmatrix} = \begin{pmatrix} 1 - \alpha\overline{\alpha} & \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 - \alpha\overline{\alpha} & -\overline{\alpha} \end{pmatrix} \begin{pmatrix} (1 - \alpha\overline{\alpha})^{-1} & \\ & 1 \end{pmatrix}.$$

It follows that

$$O^{S'}({}^t s', \widetilde{f}') = \eta_v(1 - \alpha\overline{\alpha})O^{S'}(s', \widetilde{f}') = \eta_v(1 - \alpha\overline{\alpha})O^{G'}(x, f').$$

This in particular implies that $f'^\dagger \in \mathcal{S}(G'(F_v))_0$.

Suppose x matches some $y \in G(F_v)$. This is equivalent to $1 - (\alpha\overline{\alpha})^{-1} \in \epsilon N \operatorname{GL}_n(E_v)$, which implies $\eta_v(1 - \alpha\overline{\alpha}) = \eta_v(-1)^n \epsilon_{D_v}^n$. Then

$$O^{G'}(x, f'^\dagger) = O^{G'}(x, \eta_v(-1)^n \epsilon_{D_v}^n f')$$

and the lemma follows. □

3.5. Multipliers.

To analyze the spectral side without truncation, we need to make use of the techniques from [BPLZZ21]. Recall that for any (complex) algebra $\mathcal{A}$, a multiplier is linear map $\mu\star : \mathcal{A} \to \mathcal{A}$ that commutes with both the left and right multiplication in $\mathcal{A}$. The space of multipliers of $\mathcal{A}$ is denoted by $Mul(\mathcal{A})$.

Let v be an archimedean place of F, $\mathfrak{t}_v$ the (complexified) Cartan subalgebra of $G(F_v)$, $\mathfrak{t}_v^*$ the dual space, and $\mathcal{Z}_{G(F_v)} \simeq \mathbb{C}[\mathfrak{t}_v]^{W_v}$ the center of the universal enveloping algebra of $G(F_v)$. Write $\chi_v = (\chi_1, \chi_2)$ then $\chi_1\chi_2$ as a character of $F_v^\times$ defines an element $\underline{a}_v = (a_v, \cdots, a_v) \in \mathfrak{t}_v^*$.

In [BPLZZ21, Definition 2.8(3)], an algebra $\mathcal{M}_v$ of holomorphic functions on $\mathfrak{t}_v^*$ is introduced (the notation in [BPLZZ21] is $\mathcal{M}_\theta^\sharp(\mathfrak{h}_\mathbb{C}^*)^W$). We do not need the precise definition of this space, but only the following property, cf. [BPLZZ21, Theorem 2.14]. There is an algebra homomorphism

$$\mathcal{M}_v \to Mul(\mathcal{S}(G(F_v))), \quad \mu \mapsto \mu\star$$

such that if σ is an irreducible representation of $G(F_v)$, and $f \in \mathcal{S}(G(F_v))$ then

$$\sigma(\mu \star f) = \mu(\lambda_\sigma)\sigma(f)$$

where λ_σ is the infinitesimal character of σ. Let ι_v be the involution on $\mathcal{M}_v$ such that $\iota_v(\mu)(z) = \mu(-\underline{a}_v - z)$ for all $z \in \mathfrak{t}_v^*$ and $\mathcal{M}_v^+$ be the subspace consisting of elements invariant under this involution. Recall that we have an involution $f \mapsto f^\vee \chi_1\chi_2$ on $\mathcal{S}(G(F_v))$ and the space $\mathcal{S}(G(F_v))^+$ consisting of functions invariant under this involution. If $f \in \mathcal{S}(G(F_v))^+$ and $\iota_v(\mu) = \mu$, then $\mu \star f \in \mathcal{S}(G(F_v))^+$, i.e.

$$(3.5) \qquad (\mu \star f)^\vee \chi_1\chi_2 = \mu \star f.$$

To see this, we only need to check that the action of both sides on any irreducible representation σ of $G(F_v)$ coincide. The left hand side equals

$$(\sigma \otimes \chi_1\chi_2)(\mu \star f)^\vee = \mu(-\lambda_{\sigma\otimes\chi_1\chi_2})(\sigma \otimes \chi_1\chi_2)^\vee(f) = \mu(-\underline{a}_v - \lambda_\sigma)(\sigma \otimes \chi_1\chi_2)^\vee(f).$$

Since $\iota_v(\mu) = \mu$ we have $\mu(-\underline{a}_v - \lambda_\sigma) = \mu(\lambda_\sigma)$. Moreover $f^\vee\chi_1\chi_2 = f$ implies $(\sigma\otimes\chi_1\chi_2)^\vee(f) = \sigma(f)$. This proves (3.5).

Put $\mathcal{Z}_G = \prod_{v|\infty} \mathcal{Z}_{G(F_v)}$, $\lambda = \otimes_{v|\infty}\lambda_v$ a character of $\mathcal{Z}_G$, $\mathcal{M} = \prod_{v|\infty} \mathcal{M}_v$ and $\mathcal{M}^+ = \prod_{v|\infty} \mathcal{M}_v^+$.

Let S be a finite set of finite places of F such that if $v \notin \mathsf{S}$ then E_v/F_v, π_v and χ_v are all unramified. Let T_0 be the set of nonsplit finite places of F and $\mathsf{T} = \mathsf{T}_0 \cup \mathsf{S}$. Let

$$\mathcal{H}_G^\mathsf{T} = \bigotimes_{v\notin\mathsf{T}} \mathcal{H}_v = \bigotimes_{v\notin\mathsf{T}} C_c^\infty(G(\mathfrak{o}_{F_v})\backslash G(F_v)/G(\mathfrak{o}_{F_v}))$$

be the spherical Hecke algebra away from T. We fix an open compact subgroup $K = \prod_{v\nmid\infty} K_v$ such that if $v \notin \mathsf{S}$ then $K_v = G(\mathfrak{o}_{F_v})$.

All the above objects for G have their counterparts for G'. For each archimedean place v we have an algebra of holomorphic functions $\mathcal{M}_v'$ which is identified with $\mathcal{M}_v \otimes \mathcal{M}_v$ and we put $\mathcal{M}_v'^+ = \mathcal{M}_v^+ \otimes \mathcal{M}_v^+$. Moreover put $\mathcal{M}' = \prod_{v|\infty} \mathcal{M}_v$ and $\mathcal{M}'^+ = \prod_{v|\infty} \mathcal{M}_v^+$. We have $\mathcal{S}(G'(F_v))^+ = \mathcal{S}(G(F_v))^+ \otimes \mathcal{S}(G(F_v))^+$, and we put $\mathcal{S}(G'(F_\infty))^+ = \prod_{v|\infty} \mathcal{S}(G'(F_v))^+$. We also have the center

of the universal enveloping algebra $\mathcal{Z}_{G'}$ which is identified with $\mathcal{Z}_G \otimes \mathcal{Z}_G$, and the spherical Hecke algbra away from $\mathtt{T}$

$$\mathcal{H}_{G'}^{\mathtt{T}} = \bigotimes_{v \notin \mathtt{T}} \mathcal{H}_{G',v} \simeq \mathcal{H}^{\mathtt{T}} \otimes \mathcal{H}^{\mathtt{T}}.$$

There is a base change homomorphism

$$\mathrm{bc} : \mathcal{Z}_{G'} \otimes \mathcal{H}_{G'}^{\mathtt{T}} \to \mathcal{Z}_G \otimes \mathcal{H}^{\mathtt{T}}$$

which is given by the usual multiplication in $\mathcal{Z}_G$ and $\mathcal{H}^{\mathtt{T}}$. We fix an open compact subgroup $K' = \prod_{v \nmid \infty} K_v'$ such that if $v \notin \mathtt{S}$ then $K_v' = G'(\mathfrak{o}_{F_v})$.

Let $\lambda = (\lambda_\infty, \lambda^{\infty,\mathtt{T}})$ be the character of $\mathcal{Z}_G \otimes \mathcal{H}_G^{\mathtt{T}}$ attached to π, and $L_0^2(G(F)\backslash G(\mathbb{A}_F)/K, \omega)[\lambda]$ be the maximal quotient of $L_0^2(G(F)\backslash G(\mathbb{A}_F)/K, \omega)$ on which $\mathcal{Z}_G \otimes \mathcal{H}_G^{\mathtt{T}}$ acts by λ. Then $\lambda' = \lambda \circ \mathrm{bc} = (\lambda, \lambda)$ is the character of $\mathcal{Z}_{G'} \otimes \mathcal{H}_{G'}^{\mathtt{T}}$ attached π_E, cf. [AC89, Chapter 1, Section 5]. We let $L_0^2(G'(F)\backslash G'(\mathbb{A}_F)/K', \omega)[\lambda']$ be the maximal quotient of $L_0^2(G'(F)\backslash G'(\mathbb{A}_F)/K', \omega)$ on which $\mathcal{Z}_{G'} \otimes \mathcal{H}_{G'}^{\mathtt{T}}$ acts by λ'. We have

$$L_0^2(G(F)\backslash G(\mathbb{A}_F), \omega)[\lambda] = \pi \oplus (\pi \otimes \eta), \quad L_0^2(G'(F)\backslash G'(\mathbb{A}_F), \omega')[\lambda'] = \pi_E.$$

The second equality is a direct consequence of [Ram]. The first needs a little explanation. Indeed let σ be an irreducible component of $L_0^2(G(F)\backslash G(\mathbb{A}_F), \omega)[\lambda]$. Then the base change π_E and σ_E to $\mathrm{GL}_{2n}(\mathbb{A}_E)$ are isobaric automorphic representations and they agree on almost all places of E of degree one over F. Therefore $\pi_E = \sigma_E$ by [Ram] and in particular σ_E is cuspidal. Let σ_0 be the Jacquet–Langlands transfer σ to $\mathrm{GL}_{2n}(\mathbb{A}_F)$. Then by [AC89, Chapter 3, Theorem 4.2(d)], either $\pi_0 = \sigma_0$ or $\pi_0 = \sigma_0 \otimes \eta$. Since Jacquet–Langlands transfer is an injective map, we conclude that either $\pi = \sigma$ or $\pi = \sigma \otimes \eta$. Our claim then follows from the fact that $L_0^2(G(F)\backslash G(\mathbb{A}_F), \omega)$ is of multiplicity one.

Let

$$\mathrm{bc} : \mathcal{M}' \otimes \mathcal{H}_{G'}^{\mathtt{T}} = (\mathcal{M} \otimes \mathcal{H}_G^{\mathtt{T}}) \otimes (\mathcal{M} \otimes \mathcal{H}_G^{\mathtt{T}}) \to \mathcal{M} \otimes \mathcal{H}_G^{\mathtt{T}}$$

be the usual multiplication map.

Proposition 3.7. *There are elements $\mu' \in \mathcal{M}'^+ \otimes \mathcal{H}_{G'}^{\mathtt{T}}$ and $\mu = \mathrm{bc}(\mu') \in \mathcal{M}^+ \otimes \mathcal{H}_G^{\mathtt{T}}$ such that for all $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ and $f \in \mathcal{S}(G(\mathbb{A}_F))$ we have*

- *$R(\mu' \star f')$ maps $L^2(G'(F)\backslash G'(\mathbb{A}_F)/K', \omega')$ into π_E,*
- *$\mu'(\lambda') = \mu'(\lambda, \lambda) = 1$, which is equivalent to $\pi_E(\mu' \star f') = \pi_E(f')$ for all $f' \in \mathcal{S}(G'(\mathbb{A}_F))_{K'}$,*

and

- *$R(\mu \star f)$ maps $L^2(G(F)\backslash G(\mathbb{A}_F)/K, \omega)$ into $\pi \oplus (\pi \otimes \eta)$,*
- *$\pi(\mu \star f) = \pi(f)$ and $(\pi \otimes \eta)(\mu \star f) = (\pi \otimes \eta)(f)$ for all $f \in \mathcal{S}(G(\mathbb{A}_F))_K$.*

Proof. Since $G' = \mathrm{Res}_{E/F}\,\mathrm{GL}_{2n}$, there is no CAP automorphic representation of $G'(\mathbb{A}_F)$ in the sense of [BPLZZ21, Definition 3.4]. By [BPLZZ21, Theorem 3.6] there is an element $\mu'' \in \mathcal{M}' \otimes \mathcal{H}_{G'}^{\mathtt{T}}$ satisfying the condition required for μ' in the proposition.

As $\mathcal{M}' = \prod_{w|\infty} \mathcal{M}_w$, for $\mu \in \mathcal{M}' \otimes \mathcal{H}_{G'}^{\mathsf{T}}$ we have the element $\iota_w(\mu)$ for each archimedean place w of E, by applying the involution ι_w only to the place w. Put

$$\mu' = \prod_{w|\infty} \mu'' \iota_w(\mu'') \in \mathcal{M}'^{+} \otimes \mathcal{H}_{G'}^{\mathsf{T}}.$$

We claim that μ' again satisfies conditions in the proposition. Indeed, the first condition is clear since μ'' already maps $L^2(G'(F) \backslash G'(\mathbb{A}_F)/K', \omega')$ into π_E, and moreover we have

$$\mu'(\lambda) = \prod_{w|\infty} \mu''(\lambda') \iota_w(\mu'')(\lambda') = 1,$$

since π_{E_w} satisfies $\pi_{E_w}^\vee = \pi_{E_w} \otimes \chi_1\chi_2$ if w is above the place v of F and we write $\chi_v = (\chi_1, \chi_2)$, which implies $\iota_w(\mu'')(\lambda') = \mu''(\lambda') = 1$.

Put $\mu = \mathrm{bc}(\mu')$. We claim that μ satisfies the conditions in the proposition. This is equivalent to $\mu(\lambda) = 1$ and if $\nu = (\nu_\infty, \nu^{\infty,\mathsf{T}})$ arises from an irreducible component σ of $L^2(G(F) \backslash G(\mathbb{A}_F)/K, \omega)$ and $\mu(\nu) \neq 0$, then σ is cuspidal and $\nu = \lambda$. By the definition of μ we have $\mu(\nu) = \mu'(\nu, \nu)$ where $(\nu, \nu) = \nu \circ \mathrm{bc}$ is the character of the algebra $\mathcal{Z}_{G'} \otimes \mathcal{H}_{G'}^{\mathsf{T}}$ obtained by pulling back the character ν, which is a character arising from the automorphic representation σ_E. Then $\mu'(\nu \circ \mathrm{bc}) \neq 0$ means that σ_E is cuspidal and $\nu \circ \mathrm{bc} = \lambda'$. Moreover $\mu(\lambda) = \mu'(\lambda') = 1$. $\qquad \square$

4. Proofs of the main theorems

We are now ready to prove Theorems 1.4 and 1.5, assuming Theorem 2.1 and Theorem 2.2.

4.1. **Proof of Theorem 1.4.** We keep the notation from the theorem and from Subsection 3.5. First we define the global spherical characters as follows. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ and $f \in \mathcal{S}(G(\mathbb{A}_F))$. Put

$$J_\pi(f) = \sum_\varphi P_\chi(\pi(f)\varphi) \overline{P_\chi(\varphi)},$$

where φ runs through an orthonormal basis of π. Let Π be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$ and $f' \in \mathcal{S}(G'(\mathbb{A}_F))$. Put

$$I_\Pi(f') = \sum_\varphi P'_\chi(\Pi(f')\varphi) \overline{P''_{\chi\eta}(\varphi)},$$

where again φ runs through an orthonormal basis of Π.

Recall that S is a finite set of finite places of F such that if $v \notin \mathsf{S}$ is finite, then E_v/F_v, π_v and χ_v are all unramified, T_0 is the set of nonsplit finite places of F and $\mathsf{T} = \mathsf{T}_0 \cup \mathsf{S}$. We require the finite set S to contain the place v_1. Let us now take test function $f = \otimes f_v \in \mathcal{S}(G(\mathbb{A}_F))$ and $f' = \otimes f'_v \in \mathcal{S}(G'(\mathbb{A}_F))$ as follows. If $v \notin \mathsf{S}$ and is finite, we take $f_v = \mathbf{1}_{G(\mathfrak{o}_{F_v})}$ and $f'_v = \mathbf{1}_{G'(\mathfrak{o}_{F_v})}$. If $v \in \mathsf{S}$ and $v \neq v_1$, or v is an archimedean place, we take a positive type test function $f_v \in \mathcal{S}(G(F_v))_0$ supported sufficiently close to 1, such that $J_{\pi_v}(f_v) \neq 0$. The existence of such a test function is given by Lemma 3.3 if v is not split, and is obvious if v is split. We choose $f'_v \in \mathcal{S}(G'(F_v))$ such that f'_v and f_v match in the sense of Lemma 2.3. If v is archimedean, since $\pi_v^\vee \otimes \chi_1\chi_2 \simeq \pi_v$ if

$\chi_v = (\chi_1, \chi_2)$, we may further assume that $f'_v \in \mathcal{S}(G'(F_v))^+$ and hence $f_v \in \mathcal{S}(G(F_v))^+$. With this additional condition, $f_v = \mathrm{bc}_v(f'_v)$. If $v = v_1$, we may assume that f_{v_1} is supported in the elliptic locus, and this is possible by Proposition 3.4.

We have constructed multipliers $\mu' \in \mathcal{M}'^+ \otimes \mathcal{H}_{G'}^{\mathtt{T}}$ and $\mu \in \mathcal{M}^+ \otimes \mathcal{H}_G^{\mathtt{T}}$ in Proposition 3.7. The key point is that these multipliers are in the "plus" subspaces. We now use the test functions $\mu' \star f'$ and $\mu \star f$. We claim that they still match. To see this we write $f' = f'^{\mathtt{T}} \otimes f'_{\mathtt{T}}$ where

$$f'^{\mathtt{T}} \in \mathcal{S}(G'(F_\infty))^+ \otimes \mathcal{H}_{G'}^{\mathtt{T}}, \quad f_{\mathtt{T}} \in \mathcal{S}(G(F_\infty))^+ \otimes \mathcal{H}_G^{\mathtt{T}},$$

and $f = f^{\mathtt{T}} \otimes f_{\mathtt{T}}$ similarly. The function $\mu' \star f'$ (resp. $\mu \star f$) is obtained from f' (resp. f) by modifying only finite places not in $\mathtt{T}$ and the archimedean places. Thus $f'_{\mathtt{T}}$ and $f_{\mathtt{T}}$ match (i.e. each local components match). Moreover we have

$$\mathrm{bc}(\mu' \star f'^{\mathtt{T}}) = \mathrm{bc}(\mu') \star \mathrm{bc}(f'^{\mathtt{T}}) = \mu \star f^{\mathtt{T}}.$$

The last equality follows from the definition of μ and $f^{\mathtt{T}}$. It follows from Lemma 2.4 and Proposition 2.6 that $\mu' \star f'^{\mathtt{T}}$ and $\mu \star f^{\mathtt{T}}$ match. This proves the claim.

With this choice of the test functions, we conclude that

$$(4.1) \qquad I_{\pi_E}(f') = J_\pi(f) + J_{\pi \otimes \eta}(f),$$

Since the functions are of positive type we conclude that $J_\pi(f) > 0$ and $J_{\pi \otimes \eta}(f) \geq 0$. Therefore $I_{\pi_E}(f') \neq 0$ and the assertion on the L-functions follows from Lemma 3.2.

We now move to the assertions on the local components of π. Let us fix a nonsplit place v of F. Since $L(s, \pi_0, \wedge^2 \otimes \chi_v|_{F_v^\times})$ has a simple pole at $s = 1$, we conclude that the Langlands parameter of $\pi_{0,v}$ takes value in $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude character $\chi_v|_{F_v^\times}$.

It remains to calculate the local root number $\epsilon(\pi_{0,E,v} \otimes \chi_v)$. By Lemma 3.6, $\epsilon_{D_v}^n \eta_v(-1)^n f_v$ and $f'^\dagger_v$ also match. We define

$$f'^\dagger = \otimes_{w \neq v} f'_w \otimes f'^\dagger_v.$$

Then by Lemma 3.5 we have

$$\chi_v(-1)^n \epsilon(\pi_{0,E,v} \otimes \chi_v) I_{\pi_E}(f') = \epsilon_{D_v}^n \eta_v(-1)^n (J_\pi(f) + J_{\pi \otimes \eta}(f)) \neq 0.$$

It follows that

$$\epsilon(\pi_{0,E,v} \otimes \chi_v) = \epsilon_{D_v}^n \eta_v(-1)^n \chi_v(-1)^n.$$

This finishes the proof of Theorem 1.4.

4.2. **Proof of Theorem 1.5.** We now move to Theorem 1.5. The technical part is, unlike the situation in Theorem 1.4 where all test functions supported in a small neighbourhood of $1 \in G(F_v)$ where v is a nonsplit place of F can be transferred to $G'(F_v)$, not all test functions f' can be transferred to $G(F_v)$. In order for the test function on $G'(F_v)$ to have a matching function f, there is a nontrivial vanishing condition on the orbital integrals. Lacking a good understanding

23

of representation theory and harmonic analysis on $S'(F_v)$, the result we obtain is unfortunately limited to the elliptic case.

We keep the notation of the theorem. Let us put $\Pi = \pi_{0,E}$. Assume that n is odd. Assume also that P'_χ and $P''_{\chi\eta}$ are not identically zero on Π. The later two conditions in particular implies that $\Pi^\vee \otimes (\chi\chi^c)^{-1} = \Pi$ and $\Pi^c = \Pi$.

Lemma 4.1. *Assume that Π_v is elliptic. Let D be the quaternion algebra over F_v, split (resp. nonsplit) if $\epsilon(\Pi_v \otimes \chi_v) = \eta_v(-1)\chi_v(-1)$ (resp. $-\eta_v(-1)\chi_v(-1)$). Then there is an $f' \in \mathcal{S}(G'(F_v))_0$ (for the group $G(F_v)$ given by this quaternion algebra D) such that $I_{\Pi_v}(f') \neq 0$.*

Proof. Let f' be a test function supported in the elliptic locus such that $I_{\Pi_v}(f') \neq 0$. Let $f'' = f' + \chi_v(-1)\epsilon(\Pi_v \otimes \chi_v)f'^\dagger$ where $-^\dagger$ is the involution defined by (3.3) in Subsection 3.4. Then $I_{\Pi_v}(f'') = 2I_{\Pi_v}(f') \neq 0$ by Lemma 3.5.

It remains to explain that $f'' \in \mathcal{S}(G'(F_v))_0$. We need to prove that if $x \in G'(F_v)$ is an elliptic element that does not match any $y \in G(F_v)$, then $O^{G'}(x, f'') = 0$.

The element x matches some $y \in G(F_v)$ if and only if $1 - (\alpha\bar\alpha)^{-1} \in \epsilon N \operatorname{GL}_n(E_v)$ where $\epsilon = \epsilon(\Pi_v \otimes \chi_v)\eta_v(-1)$ in $F_v^\times/NE_v^\times$. Since $x \in G'(F_v)$ is elliptic, $1 - (\alpha\bar\alpha)^{-1}$ is an elliptic element in $\operatorname{GL}_n(E_v)$ in the usual sense and therefore $1 - (\alpha\bar\alpha)^{-1} \in \epsilon N \operatorname{GL}_n(E_v)$ if and only if $\det(1 - \alpha\bar\alpha^{-1}) \in \epsilon^n N E_v^\times$. Since n is odd, this is further equivalent to $\eta_v(1 - (\alpha\bar\alpha)^{-1}) = \epsilon$, which simplifies to $\eta_v(1 - \alpha\bar\alpha) = \epsilon(\Pi_v \otimes \chi_v)\chi_v(-1)$.

Therefore if x does not match any $y \in G(F_v)$, then $\eta_v(1 - \alpha\bar\alpha) = -\epsilon(\Pi_v \otimes \chi_v)\chi_v(-1)$. We have

$$O^{G'}(x, f'^\dagger) = O^{S'}({}^t s', \widetilde{f'}) = \eta_v(1 - \alpha\bar\alpha)O^{S'}(s', \widetilde{f'}) = -\epsilon(\Pi_v \otimes \chi_v)\chi_v(-1)O^{G'}(x, f')$$

which implies $O^{G'}(x, f'') = 0$. Note that the second equality follows from the calculations in the proof of Lemma 3.6. $\square$

A similar argument also proves the following lemma.

Lemma 4.2. *Let D be the quaternion algebra over F_v, split (resp. nonsplit) if $\epsilon(\Pi_v \otimes \chi_v) = \chi_v(-1)\eta_v(-1)$ (resp. $-\chi_v(-1)\eta_v(-1)$). Assume that Π_v is supercuspidal. Let $f' \in \mathcal{S}(G'(F_v))$ and*

$$f'_{\Pi_v}(g) = \sum_W \langle \Pi_v(f_v)W, \Pi_v(g)W \rangle$$

where W runs through an orthonormal basis of Π_v. If $f'' \in \mathcal{S}(G'(F_v))$ such that

$$\int_{Z'(F_v)} f''(zg)\omega_{\Pi_v}(z)\mathrm{d}z = f'_{\Pi_v}(g),$$

then $f'' \in \mathcal{S}(G'(F_v))_0$ (for the group $G(F_v)$ given by this quaternion algebra D).

Proof. We keep the notation from the proof of Lemma 3.5 and 4.1. To simplify notation, if f' is a matrix coefficient of Π_v, by the orbital integral $O(x, f')$ we mean $O(x, f'_1)$ where $f'_1 \in \mathcal{S}(G'(F_v))$ is a function such that

$$\int_{Z'(F_v)} f'_1(zg)\omega_{\Pi_v}(z)\mathrm{d}z = f'(g)$$

24

This is independent from the choice of f_1'.

The Hom-spaces $\mathrm{Hom}_{H'(F)}(\Pi_v \otimes \chi_{H',v}, \mathbb{C})$ and $\mathrm{Hom}_{H''(F_v)}(\Pi_v \otimes \chi_v, \mathbb{C})$ are both of dimension one, and for a fixed regular semisimple $x \in G'(F_v)$, the linear form

$$(W, W') \mapsto O^{G'}(x, \langle W, \Pi_v(\cdot)W' \rangle)$$

defines an element in

$$\mathrm{Hom}_{H'(F)}(\Pi_v \otimes \chi_{H',v}, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{H''(F_v)}(\Pi_v \otimes \chi_v, \mathbb{C})}.$$

It follows that there is a function $A(x)$ on $G'(F_v)$ independent of W and W' such that

$$O^{G'}(x, \langle W, \Pi_v(\cdot)W' \rangle) = A(x)\ell'(W)\overline{\ell''(W')}.$$

Thus

$$O^{G'}(x, f'_{\Pi_v}) = A(x)I_{\Pi_v}(f').$$

Let us now consider $O^{G'}(x, (f'^\dagger)_{\Pi_v})$. By the proof of Lemma 3.5, on the one hand, we have

$$O^{G'}(x, (f'^\dagger)_{\Pi_v}) = A(x)I_{\Pi_v}(f'^\dagger) = \epsilon(\Pi_v \otimes \chi_v)A(x)I_{\Pi_v}(f') = \epsilon(\Pi_v \otimes \chi_v)O^{G'}(x, f'_{\Pi_v}).$$

On the other hand by the proof Lemma 4.1 we have

$$O^{G'}(x, (f'^\dagger)_{\Pi_v}) = O^{G'}(x, (f'_{\Pi_v})^\dagger) = \eta_v(1 - \alpha\overline{\alpha})O^{G'}(x, f'_{\Pi_v}).$$

Thus if $O^{G'}(x, f'_{\Pi_v}) \neq 0$ we have $\epsilon(\Pi_v \otimes \chi_v) = \eta_v(1 - \alpha\overline{\alpha})$. As in the proof of Lemma 4.1, this implies that $f'' \in \mathcal{S}(G'(F_v))_0$ (for the group G given by this quaternion algebra D). $\qquad \square$

Proof of Theorem 1.5. We just need to reverse the argument in the proof of Theorem 1.4. Let D be the quaternion algebra over F that splits at all $v \notin \Sigma$ and split places, and at a nonsplit place $v \in \Sigma$, it is split (resp. nonsplit) if $\epsilon(\Pi_v \otimes \chi_v) = \chi_v(-1)\eta_v(-1)$ (resp. $-\chi_v(-1)\eta_v(-1)$). Let $G = \mathrm{GL}_n(D)$.

By the assumption of the theorem we know that P'_χ and $P''_{\chi\eta}$ are not identically zero on Π. This implies that if v is a split place of F and w a place of E above it, then $\Pi_w \simeq \pi_{0,v}$ and $\Pi_w^\vee \otimes \chi_1\chi_2 \simeq \Pi_w$ if we write $\chi_v = (\chi_1, \chi_2)$.

The main technical assumption of the theorem is the following. There is a finite set Σ of places of F containing dyadic places, such that if $v \notin \Sigma$, then E_v/F_v, $\pi_{0,v}$ and χ_v are all unramified. Moreover if $v \in \Sigma$, then either v splits in E or π_{0,E_v} is supercuspidal, and there is at least one place v_1 in Σ such that $\pi_{0,E_{v_1}}$ is elliptic. As in the proof of Theorem 1.4 we have the sets of places $\mathtt{S}$ and $\mathtt{T}$. We require that the set $\mathtt{S}$ contains Σ.

We choose the test function f' as follows. Assume first that $v \neq v_1$. If $v \notin \Sigma$, then we choose $f'_v = \mathbf{1}_{G'(\mathfrak{o}_{F_v})}$ and $f_v = \mathbf{1}_{G(\mathfrak{o}_{F_v})}$. If v is infinite then we choose $f'_v \in \mathcal{S}(G'(F_v))^+$ such that $I_{\Pi_v}(f'_v) \neq 0$ and let $f_v = \mathrm{bc}_v(f'_v) \in \mathcal{S}(G(F_v))^+$ be the test function that matches it. This is possible because $\Pi_w^\vee \otimes \chi_1\chi_2 \simeq \Pi_w$. If $v \in \Sigma$ and is split we choose any f'_v such that $I_\Pi(f'_v) \neq 0$ and let f_v be the

function that matches f'_v in the sense of Lemma 2.3. If $v \in \Sigma$ is nonsplit and Π_v is supercuspidal, then we take an $f''_v \in \mathcal{S}(G'(F_v))$ and choose $f'_v \in \mathcal{S}(G'(F_v))$ such that

$$\int_{Z'(F_v)} f'_v(zg)\omega_{\Pi_v}(z)\mathrm{d}z = (f''_v)_{\Pi_v}(g),$$

as in Lemma 4.2. We may assume that $I_{\Pi'_v}(f'_v) \neq 0$. By Lemma 4.2, $f'_v \in \mathcal{S}(G'(F_v))_0$ and we let $f_v \in \mathcal{S}(G(F_v))$ be a test functions that matches f'_v. If $v = v_1$, then we choose f'_v to be a test function supported in the elliptic locus such that $I_{\Pi_v}(f'_v) \neq 0$ and $f'_v \in \mathcal{S}(G'(F_v))_0$. By Lemma 4.1 such a test function exists. We let $f_v \in \mathcal{S}(G(F_v))$ be a test functions that matches f'_v.

For this test function we have $I_\Pi(f') \neq 0$. Now argue as in the proof of Theorem 1.4, we conclude that

(4.2) $$I_\Pi(f') = J_\pi(f) + J_{\pi \otimes \eta}(f).$$

Here π is an irreducible cuspidal representation of $G(\mathbb{A}_F)$ such that $\pi_E = \Pi$. So P_χ is not identically on either on π or $\pi \otimes \eta$. But as a character of $G(\mathbb{A}_F)$, η is trivial when restricted to $H(\mathbb{A}_F)$. It follows that P_χ is identically zero on π if and only if it is so on $\pi \otimes \eta$. Thus it is not identically zero on both. Finally π_0, π and $\pi \otimes \eta$ agree at all the split places. So either π or $\pi \otimes \eta$ is the Jacquet–Langlands transfer of π_0 to $G(\mathbb{A}_F)$.

By Theorem 1.4, the quaternion algebra D satisfies $\epsilon(\pi_{0,E,v} \otimes \chi_v) = \epsilon_{D_v}^n \chi_v(-1)^n \eta_v(-1)^n$ at all place v. As n is odd, this determines D uniquely. This finishes the proof of Theorem 1.5. $\square$

5. Analysis of the orbits

5.1. **Semisimple elements in S'.** In this section, E/F is a quadratic field extension of either number fields or local fields of characteristic zero. Recall that we have the groups $G' = \mathrm{Res}_{E/F} \mathrm{GL}_{2n}$, $H' = \mathrm{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_n)$ embedded in G' as diagonal blocks, $H'' = \mathrm{GL}_{2n,F}$, and $Z' \simeq \mathrm{GL}_{1,F}$ the split center of G'. We also have a symmetric space

$$S' = \{g\bar{g}^{-1} \mid g \in G'\}$$

on which H' acts by twisted conjugation. An element in $S'(F)$ is called semisimple if its H'-orbit is (Zariski) closed. It is called regular semisimple if its stabilizer in H' is a torus of dimension n. It is called elliptic if further more its stabilizer in H' is an elliptic torus modulo Z' (over F). An element $g \in G'(F)$ is semisimple (resp. regular semisimple, resp. elliptic) if $g\bar{g}^{-1}$ is so in $S'(F)$.

In what follows we are going to make repeated use of the following lemma without mentioning it, cf. [AC89, Chapter 1, Lemma 1.1].

Lemma 5.1. *Let $g \in \mathrm{GL}_n(E)$. Then Ng is conjugate to an element in $\mathrm{GL}_n(F)$. Moreover if $g_1, g_2 \in \mathrm{GL}_n(E)$, then g_1 and g_2 are twisted conjugate if and only if Ng_1 and Ng_2 are conjugate in $\mathrm{GL}_n(E)$.*

We first classify all semisimple elements in $S'(F)$.

Lemma 5.2. *Every semisimple element in $S'(F)$ is in the $H'(F)$-orbit of the form $s'(\alpha, n_1, n_2, n_3)$, where $n_1 + n_2 + n_3 = n$ is a partition of n, $\alpha \in \mathrm{GL}_{n_1}(E)$, $\alpha\overline{\alpha} \in \mathrm{GL}_{n_1}(F)$ is semisimple in the usual sense, $\det(\alpha\overline{\alpha} - 1) \neq 0$, and*

$$
s'(\alpha, n_1, n_2, n_3) = \begin{pmatrix}
\alpha & & & 1_{n_1} & & \\
& 0_{n_2} & & & 1_{n_2} & \\
& & 1_{n_3} & & & 0_{n_3} \\
1_{n_1} - \alpha\overline{\alpha} & & & -\overline{\alpha} & & \\
& 1_{n_2} & & & 0_{n_2} & \\
& & 0_{n_3} & & & 1_{n_3}
\end{pmatrix}
$$

Proof. Let $s' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a semisimple element. We claim that $A\overline{A}$, $D\overline{D}$, $\overline{B}C$ and $C\overline{B}$ are all semisimple elements in $M_n(E)$ in the usual sense. To see this we may assume that F is algebraically closed as being semisimple is a property that does not depend on the base field. Then $E = F \times F$, S' consists of elements of the form (g, g^{-1}), $g \in \mathrm{GL}_{2n}(F)$, and it is identified with $\mathrm{GL}_n(F)$ by projection to the first factor. The group H' is identified with four copies of $\mathrm{GL}_n(F)$, and two copies of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ respectively acts on $\mathrm{GL}_{2n}(F)$ by left and right translation. The claim then reduces to [JR96, Lemma 4.2].

We will use twisted conjugation by elements in $H'(F)$ to reduce s' to an element of the form $s'(\alpha, n_1, n_2, n_3)$. This takes several steps. When we say that "s' or one of the blocks in s' takes a particular form", or "we may assume a block of s' is of the form", we mean that after replacing s' by its twisted conjugation by elements in $H'(F)$ which do not change the particular shape of s' we have achieved in the previous steps, that block of s' takes the shape that we want.

Step 1: *Simplifying B.* We may assume that B is of the form

$$
\begin{pmatrix} 1_{n-n_3} & \\ & 0_{n_3} \end{pmatrix}.
$$

Make a partition

$$
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}
$$

of the matrix A with $A_1 \in M_{n-n_3}(E)$, and similar partitions for C and D. From the condition that $s'\overline{s'} = 1$, we conclude

$$
A \begin{pmatrix} 1_{n-n_3} & \\ & 0_{n_3} \end{pmatrix} + \begin{pmatrix} 1_{n-n_3} & \\ & 0_{n_3} \end{pmatrix} \overline{D} = 0, \quad C\overline{A} + D\overline{C} = 0,
$$

and

$$
A\overline{A} + \begin{pmatrix} 1_{n-n_3} & \\ & 0_{n_3} \end{pmatrix} \overline{C} = 1_n, \quad C \begin{pmatrix} 1_{n-n_3} & \\ & 0_{n_3} \end{pmatrix} + D\overline{D} = 1_n.
$$

27

It follows that $A_3 = 0$, $D_2 = 0$, $A_4\overline{A_4} = D_4\overline{D_4} = 1_{n_3}$. Thus we may assume $A_4 = D_4 = 1_{n_3}$. Thus s' takes the following form

$$s' = \begin{pmatrix} A_1 & A_2 & 1 & \\ & 1 & & 0 \\ C_1 & C_2 & -\overline{A_1} & \\ C_3 & C_4 & D_3 & 1 \end{pmatrix}.$$

Since $A\overline{A}$ is semisimple in the usual sense, $A_1\overline{A_1}$ is so. We may further assume that $A_1\overline{A_1}$ has entries in F. Then C_1 has entries in F.

Step 2: C_1 is invertible. To see this, first as $\overline{B}C$ is semisimple in the usual sense, we conclude that $C_1 \in M_{n-n_3}(F)$ is semisimple in the usual sense. If C_1 is not invertible we may assume that C_1 is of the form $\begin{pmatrix} 0 & \\ & C_1' \end{pmatrix}$ where C_1' is invertible. Using the fact that $\overline{B}C$ and $C\overline{B}$ are semisimple in the usual sense, we conclude that C has to be of the form

$$\begin{pmatrix} 0 & & \\ & C_1' & * \\ & * & * \end{pmatrix}.$$

Therefore

$$\overline{A_1}A_1 = \begin{pmatrix} 1 & \\ & 1 - C_1' \end{pmatrix}.$$

Since 1 is not an eigenvalue of $1 - C_1'$, we may assume that A_1 takes the form $\begin{pmatrix} 1 & \\ & A_{12} \end{pmatrix}$. It follows that A takes the form

$$A = \begin{pmatrix} 1 & & A_{21} \\ & A_{12} & A_{22} \\ & & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} A_{21} \\ A_{22} \end{pmatrix}.$$

That the upper right corner of C equals zero implies that the upper right corner of A is purely imaginary. Now twisted conjugate s' by an element in $H'(F)$ of the form

$$\left(\begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ * & & 1 \end{pmatrix} \right)$$

we may assume that the upper right corner is zero.

Then it follows that s' is of the form

$$
\begin{pmatrix}
1 & & 1 & & & \\
& * & * & 1 & & \\
& & 1 & & 0 & \\
0 & & & 1 & & \\
& C_1' & * & & * & \\
& * & * & & * & 1
\end{pmatrix}.
$$

For any $\lambda \in F^{\times}$, the element

$$
\begin{pmatrix}
1 & & \lambda & & & \\
& * & * & 1 & & \\
& & 1 & & 0 & \\
0 & & & 1 & & \\
& C_1' & * & & * & \\
& * & * & & * & 1
\end{pmatrix}
=
\begin{pmatrix}
\lambda & & & & & \\
& * & & & & \\
& & * & & & \\
& & & 1 & & \\
& & & & * & \\
& & & & & *
\end{pmatrix}
\begin{pmatrix}
1 & & 1 & & & \\
& * & * & 1 & & \\
& & 1 & & 0 & \\
0 & & & 1 & & \\
& C_1' & * & & * & \\
& * & * & & * & 1
\end{pmatrix}
\begin{pmatrix}
\lambda & & & & & \\
& * & & & & \\
& & * & & & \\
& & & 1 & & \\
& & & & * & \\
& & & & & *
\end{pmatrix}^{-1}
$$

is in the same $H'(F)$-orbit of s'. Since s' is semisimple, it follows that the limit as $\lambda \to 0$

$$
\begin{pmatrix}
1 & & 0 & & & \\
& * & * & 1 & & \\
& & 1 & & 0 & \\
0 & & & 1 & & \\
& C_1' & * & & * & \\
& * & * & & * & 1
\end{pmatrix},
$$

is also in the orbit. It is a contradiction, since the rank of B is a constant in an $H'(F)$-orbit. This proves the claim that C_1 is invertible.

$\underline{\textit{Step 3: final simplification.}}$ Once we have that C_1 is invertible, we may assume that C_2 and C_3 are both zero. Then by $C\overline{A} + D\overline{C} = 0$, we conclude that A_2 and D_3 are zero. Using semisimplicity of s' again we conclude that C_4 should be zero. So we arrive at the conclusion that s' takes the form

$$
\begin{pmatrix}
A_1 & & 1 & \\
& 1 & & 0 \\
1 - A_1\overline{A_1} & & -\overline{A_1} & \\
& & & 1
\end{pmatrix}.
$$

Finally we may replace A_1 by its twisted conjugation so that $A_1 = \begin{pmatrix} \alpha & \\ & 0 \end{pmatrix}$ where α is invertible and $\alpha\overline{\alpha} \in \mathrm{GL}_{n_1}(F)$. The lemma then follows. $\qquad \square$

Lemma 5.3. *Let $s' = s'(\alpha, n_1, n_2, n_3)$ be a semisimple element in $S'(F)$ as in Lemma 5.2. It is regular if and only if $n_1 = n$ and $\alpha\overline{\alpha} \in \mathrm{GL}_n(F)$ is regular semisimple in the usual sense. It is*

elliptic if $\alpha\overline{\alpha} \in \mathrm{GL}_n(F)$ is elliptic in the usual sense. Two regular semisimple $s'(\alpha_1, n, 0, 0)$ and $s'(\alpha_2, n, 0, 0)$ are in the same $H'(F)$-orbit if and only if α_1 and α_2 are twisted conjugate in $\mathrm{GL}_n(E)$.

Proof. The stabilizer of s' in H' is isomorphic to

$$(\mathrm{GL}_{n_1, E})_{\alpha, \text{twisted}} \times \mathrm{GL}_{n_2, E} \times (\mathrm{GL}_{n_3, F} \times \mathrm{GL}_{n_3, F}).$$

The twisted stabilizer $(\mathrm{GL}_{n_1, E})_{\alpha, \text{twisted}}$ is an inner form of $(\mathrm{GL}_{n_1, F})_{\alpha\overline{\alpha}}$, whose dimension is at least n_1. Thus the dimension of $H'_{s'}$ is at least

$$n_1 + 2n_2^2 + 2n_3^3 \geq n,$$

and the equality holds if and only if $n_2 = n_3 = 0$ and $\dim(\mathrm{GL}_{n_1, E})_{\alpha, \text{twisted}} = n$, which is equivalent to that $\alpha\overline{\alpha}$ is regular semisimple in $\mathrm{GL}_n(F)$. The other assertions of the lemma are obvious. $\square$

To simplify notation, for any $\alpha \in M_n(E)$, we put $s'(\alpha) = s'(\alpha, n, 0, 0)$.

Let $\mathbf{A}^n$ be the affine space of dimension n over F and

$$q' : S' \to \mathbf{A}^n$$

be the morphism

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \mathrm{Tr} \wedge^i (2A\overline{A} - 1), \quad i = 1, \cdots, n.$$

Lemma 5.4. *The map q' is a categorical quotient.*

Proof. This is a geometric statement, so we may assume that F is algebraically closed. Then as in the proof of Lemma 5.2 we are then reduced to the case considered in [Guo96, Zha15a]. $\square$

5.2. **Explicit étale Luna slices.** Let us begin with some general discussion. Let G be a reductive algebraic group acting on an affine algebraic variety X. We denote by $X \to X//G$, or simply $X//G$ the categorical quotient. Let $x \in X$ and T_x be the tangent space of X at x. We fix a G_x-invariant inner product on T_x. Let TO_x be the tangent space of orbit Gx at x, and $N_x = TO_x^\perp$ the orthogonal complement in T_x. The group G_x then acts on N_x, and this is called the sliced representation at x. An étale Luna slice at x is a locally closed subvariety $Z \subset X$, containing x and stable under the G_x action, together with a strongly étale G_x-equivariant morphism $\iota : Z \to N_x$ such that the morphism

$$G \times_{G_x} Z \to X$$

is strongly étale. Here if X and Y are affine varieties with G actions, a morphism $X \to Y$ is called strongly étale if the induced morphism $X//G \to Y//G$ is étale and the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X//G & \longrightarrow & Y//G \end{array}$$

is Cartesian.

We now come back to the action of the group H' on S'. Let

$$\mathfrak{s}' = \left\{ \begin{pmatrix} & X \\ Y & \end{pmatrix} \,\middle|\, X, Y \in M_n(E)^- \right\},$$

where $M_n(E)^- = \{X \in M_n(E) \mid \overline{X} = -X\}$. This is viewed as an algebraic variety over F. It is isomorphic to the tangent space of S' at 1. The stabilizer of 1 in H' is isomorphic to $\mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$, which acts on $\mathfrak{s}'$ by conjugation. Let $g \in G'(F)$ and $s' = g\overline{g}^{-1} \in S'(F)$. The tangent space of S' at s' is identified with

$$T_{s'} = \{gY\overline{g}^{-1} \mid Y \in \mathfrak{s}'\}.$$

The tangent space of the H'-orbit of s' is identified with a subspace

$$TO_{s'} = \{Xs' - s'\overline{X} \mid X \in \mathfrak{h}'\}.$$

We fix an inner product on T_s by

$$\langle gY_1\overline{g}^{-1}, gY_2\overline{g}^{-1} \rangle = \mathrm{Tr}\, Y_1 Y_2.$$

This inner product is $H'_{s'}$-invariant. Put

$$N_{s'} = \{Ys' \mid \theta'(Y) = -Y,\ Ys' = -s'\overline{Y}\}.$$

Then we have an orthogonal decomposition

$$T_{s'} = TO_{s'} \oplus^{\perp} N_{s'}.$$

Thus $N_{s'}$ is the sliced representation.

Put

$$Z' = \left\{ xs' \mid x\theta(x) = 1,\ xs' = s'\overline{x}^{-1},\ \det(1 + x) \neq 0,\ \det\left((1 - \mathrm{Ad}(xs'))|_{\mathfrak{g}'^{\perp}_{s'}} \right) \neq 0 \right\},$$

where $\mathfrak{g}'_{s'}$ stands for the Lie algebra of the centralizer of s' in G', and the orthogonal complement is taken with respect to an $H'_{s'}$-invariant inner product. Then Z' is a locally closed subscheme of S'. Put also

$$\iota' : Z' \to N_{s'}, \quad xs' \mapsto (1 - x)(1 + x)^{-1}s'.$$

Lemma 5.5. *The maps $\iota' : Z' \to N_{s'}$ and $(H' \times Z')//H'_{s'} \to S'$ are strongly étale. Therefore Z' is an étale Luna slice at s'.*

Proof. Since this is a geometric statement we may assume that the base field F is algebraically closed. As in the proof of Lemma 5.4, this lemma is then reduced to the explicit construction of étale Luna slices in the case of relative trace formula of Guo–Jacquet, and this is explained in [Zha15a, Subsection 5.3] (which in turn is based on [JR96, Section 5.2]). □

We will need more concrete descriptions of the sliced representations. Let $s' = s'(\alpha, n_1, n_2, n_3)$ be a semisimple element in $S'(F)$. Its stabilizer in H' equals

$$H_1' \times H_2' \times H_3' = (\mathrm{GL}_{n_1,E})_{\alpha,\text{twisted}} \times \mathrm{GL}_{n_2,E} \times (\mathrm{GL}_{n_3,F} \times \mathrm{GL}_{n_3,F}).$$

The sliced representation at $s'(\alpha, n_1, n_2, n_3)$ is isomorphic to $V_1' \oplus V_2' \oplus V_3'$ where

$$V_1' = \{A \in M_{n_1}(E) \mid \alpha\overline{A} = A\overline{\alpha}\}, \quad V_2' = M_{n_2}(E), \quad V_3' = M_{n_3}(E)^- \oplus M_{n_3}(E)^-.$$

In the three extreme cases where $n_1 = n$, $n_2 = n$ and $n_3 = n$, we have the following descriptions.

(1) Assume $n_1 = n$, $n_2 = n_3 = 0$. The embedding of $(\mathrm{GL}_{n,E})_{\alpha,\text{twisted}}$ in H' is given by

$$h \mapsto \begin{pmatrix} h & \\ & \overline{h} \end{pmatrix}.$$

The embedding of V_1' in $N_{s'}$ is given by

$$A \mapsto \begin{pmatrix} & A \\ -\overline{A}(1 - \alpha\overline{\alpha}) & \end{pmatrix} s'.$$

(2) Assume $n_2 = n$, $n_1 = n_3 = 0$. Then $s' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. The embedding of $\mathrm{GL}_{n,E}$ in H' is given by

$$h \mapsto \begin{pmatrix} h & \\ & \overline{h} \end{pmatrix}.$$

The embedding of $V_2' = M_n(E)$ into $N_{s'}$ is given by

$$A \mapsto \begin{pmatrix} & A \\ -\overline{A} & \end{pmatrix} s' = \begin{pmatrix} A & \\ & -\overline{A} \end{pmatrix}$$

(3) Assume $n_1 = n_2 = 0$. Then $s' = 1$. The embedding of $\mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ in H' is given by

$$(h_1, h_2) \mapsto \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}.$$

The embedding of $V_3' = M_{n_3}(E)^- \oplus M_{n_3}(E)^-$ into $N_{s'}$ is given by

$$(A, B) \mapsto \begin{pmatrix} & A \\ B & \end{pmatrix} s' = \begin{pmatrix} & A \\ B & \end{pmatrix}.$$

In general, in obvious notation, we have $Z' = Z_1' \times Z_2' \times Z_3'$ according to the decomposition $N_{s'} = V_1' \times V_2' \times V_3'$. We also have the $H_{s'}'$-equivariant morphism $\iota' = \iota_1' \times \iota_2' \times \iota_3'$.

5.3. Semisimple elements in G. Recall that D is a quaternion algebra over F with a fixed embedding $E \to D$, $G = \mathrm{GL}_n(D)$, $H = \mathrm{Res}_{E/F} \mathrm{GL}_n$ the centralizer of $E^\times$ in G, $Z \simeq \mathrm{GL}_{1,F}$ the split center of G. Recall that θ is the automorphism of G given by conjugation by $\begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}$. Then H is the stabilizer of θ.

Put $S = \{g\theta(g)^{-1} \mid g \in G\}$ and H acts on S by conjugation. Similarly to S', an element in $S(F)$ is called semisimple if its H-orbit is (Zariski) closed. It is called regular semisimple its stabilizer in H is a torus of dimension n. It is called elliptic if further more its stabilizer in H is an elliptic torus modulo Z (over F). An element $g \in G(F)$ is semisimple (resp. regular semisimple, resp. elliptic) if $g\theta(g)^{-1}$ is so in $S(F)$.

The following lemma summarizes [Guo96, Proposition 1.2].

Lemma 5.6. *Every semisimple $g \in G(F)$ is in the $(H \times H)(F)$-orbit of*

$$
g(\beta, n_1, n_2, n_3) = \begin{pmatrix}
1_{n_1} & & & \epsilon\beta & & \\
& 0_{n_2} & & & \epsilon 1_{n_2} & \\
& & 1_{n_3} & & & 0_{n_3} \\
\overline{\beta} & & & 1_{n_1} & & \\
& 1_{n_2} & & & 0_{n_2} & \\
& & 0_{n_3} & & & 1_{n_3}
\end{pmatrix},
$$

where $\beta \in \mathrm{GL}_{n_1}(E)$, $\beta\overline{\beta} \in \mathrm{GL}_n(F)$ and $\det(1 - \epsilon\beta\overline{\beta}) \neq 0$. It is regular semisimple (resp. elliptic) if $n_2 = n_3 = 0$ and $\beta\overline{\beta}$ is regular semisimple (resp. elliptic) in $\mathrm{GL}_n(F)$ in the usual sense.

If $n_2 = n_3 = 0$, we write $s(\beta, n, 0, 0) = s(\beta)$ and $g(\beta) = g(\beta, n, 0, 0)$.

Let $g \in G$ and $s = g\theta(g)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S$. Define

$$
q : G \to \mathbf{A}^n, \quad g \mapsto \mathrm{Tr} \wedge^i A, \quad i = 1, \cdots n.
$$

Lemma 5.7. *The map q is a categorical quotient.*

Proof. This is again a geometric statement, so we may assume that F is algebraically closed. Then the lemma reduces to the case considered in [Guo96, Zha15a]. $\qquad\square$

At each semisimple $g \in G(F)$, we construct an explicit étale Luna slice as follows. Put $s = g\theta(g)^{-1} \in S(F)$. We note $(H \times H)_g \simeq H_s$. The normal space of the $H \times H$-orbit of g at the point g is identified with

$$
N_g = \{Yg \mid Y \in \mathfrak{s}, Ys = sY\}.
$$

The group $(H \times H)_g$ acts on N_g and N_g is the sliced representation. Let

$$
Z = \{yg \in G \mid y\theta(y) = 1, \ ys = sy, \ \det(1 + y) \neq 0, \ \det\left((1 - \mathrm{Ad}(ys))|_{\mathfrak{g}_s^\perp}\right) \neq 0\},
$$

where $\mathfrak{g}_s$ stands for the Lie algebra of the centralizer of s in G, and the orthogonal complement is taken with respect to an H_s-invariant inner product. There is a map $\iota : Z \to N_s$ given by

$$yg \mapsto (1-y)(1+y)^{-1}g.$$

Lemma 5.8. *The maps $Z \to N_s$ and $((H \times H) \times Z)//(H \times H)_g \to G$ are strongly étale. Therefore Z is an étale Luna slice at g.*

Proof. We first recall that $(H \times H)_g = H_s$. Checking that $((H \times H) \times Z)//(H \times H)_g \to G$ is strongly étale is equivalent to checking that $(H \times Z)//H_s \to S$ is strongly étale. This is a geometric statement so we may assume that F is algebraically closed. Then the action of H on S is reduced to two copies of $\mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ acting on $\mathrm{GL}_{2n,F}$. The lemma again reduces to the description in [Zha15a, Section 5.3]. $\qquad\square$

We now give more concrete descriptions of the sliced representation at $g = g(\beta, n_1, n_2, n_3)$. Let $\mathfrak{s}$ be the space consisting of matrices of the form

$$\begin{pmatrix} & \epsilon Y \\ \overline{Y} & \end{pmatrix}, \quad Y \in M_n(E).$$

It is identified with the tangent space of S at 1, and the group H acts on $\mathfrak{s}$ by conjugation. We also write $\mathfrak{s}_n$ to indicate the size of the space $\mathfrak{s}$. The stabilizer $(H \times H)_g$ is isomorphic to $H_1 \times H_2 \times H_3$ where

$$H_1 = (\mathrm{GL}_{n_1,E})_{\beta,\mathrm{twisted}}, \quad H_2 = \mathrm{GL}_{n_2,E}, \quad H_3 = \mathrm{GL}_{n_3,E}.$$

The norm space $N_g = V_1 \oplus V_2 \oplus V_3$ with

$$V_1 = \left\{ \begin{pmatrix} & \epsilon Y \\ \overline{Y} & \end{pmatrix} \in \mathfrak{s}_{n_1} \ \middle| \ \beta\overline{Y} = Y\overline{\beta} \right\}, \quad V_2 = \mathfrak{s}_{n_2}, \quad V_3 = \mathfrak{s}_{n_3}.$$

We also have $Z = Z_1 \times Z_2 \times Z_3$ according to the decomposition $N_g = V_1 \times V_2 \times V_3$, and the H_s-equivariant morphism $\iota = \iota_1 \times \iota_2 \times \iota_3$.

6. Transfer of test functions

6.1. **Matching.** The goal of this section is to prove Theorem 2.1. Throughout this section, we suppress the subscript v and assume that E/F is a quadratic extension of local fields of characteristic zero.

Recall that we have the categorical quotients

$$q' : S' \to \mathbf{A}^n, \quad q : G \to \mathbf{A}^n.$$

Two semisimple elements $s' \in S'(F)$ and $g \in G(F)$ match if they are mapped to the same point in $\mathbf{A}^n$. More concretely we have

Lemma 6.1. *Two regular semisimple elements $s'(\alpha) \in S'(F)$ and $g(\beta) \in G(F)$ match if and only if $-(1 - \alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}$ and $\epsilon\beta\overline{\beta}$ have the same characteristic polynomial.*

34

Proof. Let $s'(\alpha) \in G'(F)$ and $g(\beta) \in G(F)$ be regular semisimple elements. The upper left $n \times n$ block of its image in $S(F)$ is

$$(1 - \epsilon\beta\overline{\beta})^{-1}(1 + \epsilon\beta\overline{\beta}).$$

By definition, that $s'(\alpha)$ and $g(\beta)$ match is equivalent to $2\alpha\overline{\alpha} - 1$ and $(1 - \epsilon\beta\overline{\beta})^{-1}(1 + \epsilon\beta\overline{\beta})$ have the same characteristic polynomial. This is further equivalent to that $-(1 - \alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}$ and $\epsilon\beta\overline{\beta}$ have the same characteristic polynomial. $\qquad\square$

We put

$$G(F)_{\mathrm{reg},0} = \left\{ g \in G(F)_{\mathrm{reg}} \;\middle|\; g\theta(g)^{-1} = \begin{pmatrix} A & \epsilon B \\ \overline{B} & \overline{A} \end{pmatrix} \in S(F), \; \tfrac{1}{2}(A+1) \in N\,\mathrm{GL}_n(E) \right\},$$

and

$$S'(F)_{\mathrm{reg},0} = \left\{ s' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S'(F)_{\mathrm{reg}} \;\middle|\; 1 - (A\overline{A})^{-1} \in \epsilon N\,\mathrm{GL}_n(E) \right\}.$$

Lemma 6.2. *The matching defines a bijection between the H-orbits in $G_{\mathrm{reg},0}$ to $S'_{\mathrm{reg},0}$.*

Proof. Take $s'(\alpha) \in S'(F)_{\mathrm{reg},0}$. By definition we can find a $\beta \in \mathrm{GL}_n(E)$ such that

$$1 - (\alpha\overline{\alpha})^{-1} = \epsilon\beta\overline{\beta}.$$

Since $\alpha\overline{\alpha} \in \mathrm{GL}_n(F)$ is regular semisimple in the usual sense, so is $\beta\overline{\beta}$. Then $g(\beta)$ is regular semisimple in $G(F)$ and matches $s'(\alpha)$ by definition. The other direction is proved similarly. $\qquad\square$

Recall from (2.5) that we have defined transfer factors on G and G'. Fixed a character $\widetilde{\eta} : E^\times \to \mathbb{C}^\times$ whose restriction to $F^\times$ equals η. Fix a purely imaginary element $\tau \in E^\times$. If $s' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in S'(F)$ be a regular semisimple element, then by Lemma 5.2, A', B', C', D' are all invertible and we define (cf. (2.5))

$$(6.1) \qquad\qquad \kappa^{S'}(s') = \chi(\tau D')\widetilde{\eta}(B').$$

If $g \in G(F)$ and $g^{-1} = \begin{pmatrix} A & \epsilon B \\ \overline{B} & \overline{A} \end{pmatrix} \in G(F)$ then by Lemma 5.6, A, B are both invertible and in (2.5) we put $\kappa^G(s) = \chi(A)$.

We also put

$$\mathcal{S}(G(F))_0 = \{ f \in \mathcal{S}(G(F)) \mid O^G(g, f) = 0 \text{ for all } g \notin G(F)_{\mathrm{reg},0} \},$$

and

$$\mathcal{S}(S'(F))_0 = \{ f' \in \mathcal{S}(S'(F)) \mid O^{G'}(s', f') = 0 \text{ for all } s' \notin S'(F)_{\mathrm{reg},0} \}.$$

Then we say that $f \in \mathcal{S}(G(F))_0$ and $f' \in \mathcal{S}(G'(F))_0$ match if

$$\kappa^G(g)O^G(g, f) = \kappa^{G'}(s')O^{G'}(s', f'_{s'}),$$

for all matching regular semisimple $g \in G(F)$ and $s' \in S'(F)$.

Theorem 6.3. *For all $f \in \mathcal{S}(G(F))_0$ there is an $f' \in \mathcal{S}(S'(F))_0$ that matches it, and vice versa.*

By (2.3) and (2.4), this theorem and Theorem 2.1 are equivalent. The rest of this section is devoted to the proof of Theorem 6.3. The basic idea is that matching of test functions is a local property, and it is enough to prove the existence of "local transfer" near each semisimple point. The orbital integrals near a semisimple point can be related to the orbital integrals on the corresponding sliced representation via semisimple descent. In the current setting the orbital integrals on the sliced representations turn out to be familiar ones. For readers' convenience we record this in the following lemma, whose proof can be found in [Zha14, Proposition 3.8].

Lemma 6.4. *Let $f \in \mathcal{S}(G(F))_0$. If for all matching semisimple $g \in G(F)$ and $s' \in S'(F)$, there is an $(H \times H)(F)$-invariant neighbourhood U of g in $G(F)$ and an $H'(F)$-invariant neighbourhood U' of s' in $S'(F)$, and a function $f'_{s'} \in \mathcal{S}(S'(F))_0$, such that for all matching regular semisimple $g \in U$ and $s' \in U'$ we have*

$$\kappa^G(g)O^G(g, f) = \kappa^{S'}(s')O^{S'}(s', f'_{s'}),$$

then there is an $f' \in \mathcal{S}(S'(F))_0$ that matches f. Similar statement holds in the other direction.

The condition in this lemma will be referred to as the existence of local transfer at g and s'. The existence of local transfer is what we will prove later in this section.

6.2. **Transfer on the sliced representations.** Near each semisimple point, the orbital integral can be connected to the orbital integral on the sliced representation of that point. As the preparation for the proof of Theorem 6.3, we explain the transfer of test functions on the sliced representations in this subsection.

The transfer of test functions on the sliced representations are closely related to the transfer of test functions in the relative trace formulae of Guo and Jacquet, which has been established in [Zha15a]. We recall it first. Put $\mathsf{V} = M_n(F) \times M_n(F)$ and $\mathsf{W} = M_n(E)$. The group $\mathsf{H}' = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ acts on V by

$$(h_1, h_2) \cdot (X_1, X_2) = (h_1 X_1 h_2^{-1}, h_2 X_2 h_1^{-1}),$$

and the group $\mathsf{H} = \mathrm{Res}_{E/F} \mathrm{GL}_{n,E}$ acts on W by twisted conjugation. The element $(X_1, X_2) \in \mathsf{V}$ is regular semisimple if $X_1 X_2 \in \mathrm{GL}_n(F)$ and is regular semisimple in the usual sense. The element $Y \in \mathsf{W}$ is regular semisimple if $Y\overline{Y} \in \mathrm{GL}_n(E)$ and is regular semisimple in the usual sense. Two regular semisimple elements $(X_1, X_2) \in \mathsf{V}$ and $Y \in \mathsf{W}$ match if $X_1 X_2$ and $\epsilon Y\overline{Y}$ have the same characteristic polynomial. For $f' \in \mathcal{S}(\mathsf{V})$ and $f \in \mathcal{S}(\mathsf{W})$, define the orbital integrals

$$O^{\mathsf{V}}((X_1, X_2), f') = \int_{\mathsf{H}_{(X_1, X_2)}(F)\backslash \mathsf{H}(F)} f'(h_1 X_1 h_2^{-1}, h_2 X_2 h_1^{-1})\eta(h_1 h_2)\mathrm{d}h_1 \mathrm{d}h_2,$$

and

$$O^{\mathsf{W}}(Y, f) = \int_{\mathsf{H}_Y(F)\backslash \mathsf{H}(F)} f(hY\overline{h}^{-1})\mathrm{d}h.$$

Put

$$\mathcal{S}(\mathsf{V})_0 = \{O^{\mathsf{V}}((X_1, X_2), f') = 0 \text{ for all regular semisimple } (X_1, X_2) \in \mathsf{V} \text{ with } X_1 X_2 \notin \epsilon N \, \mathrm{GL}_n(E)\}.$$

Put also for $(X_1, X_2) \in \mathsf{V}$,

$$\kappa^{\mathsf{V}}(X_1, X_2) = \eta(X_1).$$

Two functions $f' \in \mathcal{S}(\mathsf{V})_0$ and $f \in \mathcal{S}(\mathsf{W})$ match if

$$\kappa^{\mathsf{V}}(X_1, X_2) O^{\mathsf{V}}((X_1, X_2), f') = O^{\mathsf{W}}(Y, f)$$

for all matching (X_1, X_2) and Y. The following is [Zha15a, Theorem 5.14].

Proposition 6.5. *For any $f' \in \mathcal{S}(\mathsf{V})_0$ there is an $f \in \mathcal{S}(\mathsf{W})$ that matches it, and vice versa.*

Let us get back to the sliced representations. Let

$$s' = s'(\alpha, n_1, n_2, n_3) \in S'(F), \quad g = g(\beta, n_1, n_2, n_3) \in G(F)$$

be matching semisimple elements. Recall that the sliced representation at s' is isomorphic to

$$(H_1', V_1') \times (H_2', V_2') \times (H_3', V_3'),$$

where

$$H_1' = (\mathrm{GL}_{n_1, E})_{\alpha, \text{twisted}}, \quad H_2' = \mathrm{GL}_{n_2, E}, \quad H_3' = \mathrm{GL}_{n_3, F} \times \mathrm{GL}_{n_3, F},$$

and

$$V_1' = \{X \in M_{n_1}(E) \mid \alpha \overline{X} = X \overline{\alpha}\}, \quad V_2' = M_{n_2}(E), \quad V_3' = M_{n_3}(E)^- \oplus M_{n_3}(E)^-,$$

The sliced representation at $g \in G(F)$ is isomorphic to

$$(H_1, V_1) \times (H_2, V_2) \times (H_3, V_3),$$

where

$$H_1 = (\mathrm{GL}_{n, E})_{\beta, \text{twisted}}, \quad H_2 = \mathrm{GL}_{n_2, E}, \quad H_3 = \mathrm{GL}_{n_3, E},$$

and

$$V_1 = \left\{Y \in M_{n_1}(E) \mid \beta \overline{Y} = Y \overline{\beta}\right\}, \quad V_2 = M_{n_2}(E), \quad V_3 = M_{n_3}(E).$$

We can speak of the transfer of functions on each component.

(1) The transfer on V_1' and V_1. The group H_1' and H_1 act on V_1' and V_1 respectively by twisted conjugation. Let L be the centralizer of $\alpha \overline{\alpha}$ in $\mathrm{GL}_{n_1}(F)$. Since $-(1 - \alpha \overline{\alpha})(\alpha \overline{\alpha})^{-1}$ and $\epsilon \beta \overline{\beta}$ have the same characteristic polynomial, it is also the centralize of $\beta \overline{\beta}$. Then H_1' and H_1 are both inner forms of L. The map

$$V_1' \to \mathfrak{h}_1', \quad X \to X \overline{\alpha}$$

is an isomorphism of representations of H_1' where H_1' acts on $\mathfrak{h}_1'$ by conjugation. The map

$$V_1 \to \mathfrak{h}_1, \quad Y \mapsto Y \overline{\beta}$$

37

is an isomorphism of representations of H_1 where H_1 acts on $\mathfrak{h}_1$ by conjugation. Thus we may speak of semisimple and regular semisimple elements in V_1' and V_1, and matching between these elements. Two semisimple elements $X \in V_1'$ and $Y \in V_1$ match if $X\overline{\alpha}$ and $Y\overline{\beta}$ have the same (reduced) characteristic polynomial. We define

$$V_{1,\mathrm{reg},0}' = \{X \in V_{1,\mathrm{reg}}' \mid X \text{ matches some } Y \in V_{1,\mathrm{reg}}\},$$

and

$$V_{1,\mathrm{reg},0} = \{Y \in V_{1,\mathrm{reg}} \mid Y \text{ matches some } X \in V_{1,\mathrm{reg}}'\}.$$

For regular semisimple $X \in V_1'$, $Y \in V_1$, and test functions $f' \in \mathcal{S}(V_1')$, $f \in \mathcal{S}(V_1)$, we may define orbital integral $O^{V_1'}(X, f')$ and $O^{V_1}(Y, f)$ as usual. Define

$$\mathcal{S}(V_1')_0 = \{f' \in \mathcal{S}(V_1') \mid O^{V_1'}(X, f') = 0 \text{ for all } X \notin V_{1,\mathrm{reg},0}'\}$$

and $\mathcal{S}(V_1)_0$ similarly. Two test functions $f' \in \mathcal{S}(V_1')_0$ and $f \in \mathcal{S}(V_1)_0$ match if

$$O^{V_1'}(X, f') = O^{V_1}(Y, f)$$

for all matching regular semisimple X and Y. The transfer of test functions between inner forms of GL_n implies that for any $f' \in \mathcal{S}(V_1')_0$ there is an $f \in \mathcal{S}(V_1)_0$ that matches it and vice versa.

(2) The transfer on V_2' and V_2. Both V_2' and V_2 are isomorphic to $M_{n_2}(E)$. Regular semisimple elements $X \in V_2'$ and $Y \in V_2$ mean $X\overline{X}$ and $Y\overline{Y}$ are in $\mathrm{GL}_{n_2}(E)$ and are regular semisimple in the usual sense. We define that they match if $X\overline{X}$ and $-\epsilon Y\overline{Y}$ have the same characteristic polynomial. We define

$$V_{2,\mathrm{reg},0}' = \{X \in V_{2,\mathrm{reg}}' \mid X \text{ matches some } Y \in V_{2,\mathrm{reg}}\},$$

and

$$V_{2,\mathrm{reg},0} = \{Y \in V_{2,\mathrm{reg}} \mid Y \text{ matches some } X \in V_{2,\mathrm{reg}}'\}.$$

For regular semisimple $X \in V_2'$, $Y \in V_2$, and test functions $f' \in \mathcal{S}(V_2')$, $f \in \mathcal{S}(V_2)$, we define orbital integral $O^{V_2'}(X, f')$ and $O^{V_2}(Y, f)$ by

$$O^{V_2'}(X, f') = \int_{H_{2,X}' \backslash H_2} f'(h^{-1}X\overline{h})\chi(h^{-1}\overline{h})\mathrm{d}h,$$

and

$$O^{V_2}(Y, f) = \int_{H_{2,Y} \backslash H_2} f(h^{-1}Y\overline{h})\chi(h^{-1}\overline{h})\mathrm{d}h.$$

Define

$$\mathcal{S}(V_2')_0 = \{f' \in \mathcal{S}(V_2') \mid O^{V_2'}(X, f') = 0 \text{ for all } X \notin V_{2,\mathrm{reg},0}'\}$$

and $\mathcal{S}(V_2)_0$ similarly. Define for regular semisimple X and Y the transfer factor

$$\kappa^{V_2'}(X) = \chi(X), \quad \kappa^{V_2}(Y) = \chi(Y).$$

Two test functions $f' \in \mathcal{S}(V_2')_0$ and $f \in \mathcal{S}(V_2)_0$ match if

$$\kappa^{V_2'}(X)O^{V_2'}(X, f') = \kappa^{V_2}(Y)O^{V_2}(Y, f)$$

for all matching regular semisimple X and Y.

Let us explain that for any $f' \in \mathcal{S}(V_2')_0$ there is an $f \in \mathcal{S}(V_2)_0$ that matches it and vice versa. First by replacing f' and f by $f'\chi$ and $f\chi$, we may assume that χ is trivial. Let $f' \in \mathcal{S}(V_2')_0$. Then $O^{V_2'}(X, f')$ is the orbital integral appearing on the W-side of the transfer problem of Guo and Jacquet. Then by Proposition 6.5, applied to $\mathsf{W} = V_2'$, there is an $\widetilde{f} \in \mathcal{S}(\mathsf{V})$ such that

$$O^{V_2'}(X, f') = \kappa^{\mathsf{V}}(Z_1, Z_2)O^{\mathsf{V}}((Z_1, Z_2), \widetilde{f}),$$

for all regular semisimple $X \in V_2'$ and (Z_1, Z_2) with $Z_1 Z_2 = X\overline{X}$. Since $f' \in \mathcal{S}(V_2')_0$, the function $\widetilde{f}$ satisfies the property that $O^{\mathsf{V}}((Z_1, Z_2), \widetilde{f}) = 0$ if $Z_1 Z_2 \notin N\,\mathrm{GL}_n(E)$ or $Z_1 Z_2 \notin -\epsilon N\,\mathrm{GL}_n(E)$. Now apply Proposition 6.5 again to $\mathsf{W} = V_2$, we conclude that there is a function $f \in \mathcal{S}(V_2)$ such that

$$\kappa^{\mathsf{V}}(Z_1, Z_2)O^{\mathsf{V}}((Z_1, Z_2), \widetilde{f}) = O^{V_2}(Y, f)$$

for all regular semisimple $X \in V_2'$ and (Z_1, Z_2) with $Z_1 Z_2 = -\epsilon Y\overline{Y}$. The function f satisfies the property that $O^{V_2}(Y, f) = 0$ if $-\epsilon Y\overline{Y} \notin N\,\mathrm{GL}_n(E)$, and hence $f \in \mathcal{S}(V_2)_0$. It follows that if $X \in V_2'$ matches $Y \in V_2$, i.e. $X\overline{X}$ and $-\epsilon Y\overline{Y}$ have the same characteristic polynomial, then

$$O^{V_2'}(X, f') = O^{V_2}(Y, f).$$

The proof of the converse direction from f to f' is the same.

(3) The transfer on V_3' and V_3 is exactly the transfer problem of Guo and Jacquet. Let

$$V_{3,\mathrm{reg},0}' = \{(X_1, X_2) \mid X_1 X_2 \in \epsilon N\,\mathrm{GL}_{n_3}(E)\}.$$

For any $f' \in \mathcal{S}(V_3')$ have the orbital integral

$$O^{V_3'}((X_1, X_2), f') = \int_{H_{3,(X_1,X_2)}' \backslash H_3'} f'(h_1^{-1} X_1 h_2, h_2^{-1} X_2 h_1)\eta(h_1 h_2)\mathrm{d}h_1 \mathrm{d}h_2.$$

For $f \in \mathcal{S}(V_3)$ we have the orbital integral

$$O^{V_3}(Y, f) = \int_{H_{3,Y} \backslash H_3} f(h^{-1} Y\overline{h})\mathrm{d}h.$$

We also have

$$\mathcal{S}(V_3')_0 = \{f' \in \mathcal{S}(V_3') \mid O((X_1, X_2), f') = 0 \text{ for all } (X_1, X_2) \notin V_{3,\mathrm{reg},0}'\}.$$

The transfer factor on V_3' is given by

$$\kappa^{V_3'}((X_1, X_2)) = \widetilde{\eta}(X_1).$$

By Proposition 6.5, for any $f' \in \mathcal{S}(V_3')_0$ there is an $f \in \mathcal{S}(V_3)$ such that

$$\kappa^{V_3'}((X_1, X_2))O^{V_3'}((X_1, X_2), f') = O^{V_3}(Y, f)$$

for all matching (X_1, X_2) and Y, and vice versa.

We put

$$\mathcal{S}(N_{s'})_0 = \mathcal{S}(V_1')_0 \otimes \mathcal{S}(V_2')_0 \otimes \mathcal{S}(V_3')_0, \quad \mathcal{S}(N_g)_0 = \mathcal{S}(V_1)_0 \otimes \mathcal{S}(V_2)_0 \otimes \mathcal{S}(V_3).$$

Let us define transfer factors on $N_{s'}$ and on N_g by setting

$$\kappa^{N_{s'}}(X_1, X_2, (X_3, X_4)) = \chi(X_2)\widetilde{\eta}(X_3), \quad \kappa^{N_g}(Y_1, Y_2, Y_3) = \chi(Y_2),$$

and define orbital integrals and matching of test functions in $N_{s'}$ and N_g in the obvious way.

The above discussion can then be summarized as the following proposition.

Proposition 6.6. *For all $f' \in \mathcal{S}(N_{s'})_0$ there is an $f \in \mathcal{S}(N_g)_0$ such that for all matching $X \in N_{s',\mathrm{reg},0}$ and $Y \in N_{g,\mathrm{reg},0}$, we have*

$$\kappa^{N_{s'}}(X)O^{N_{s'}}(X, f') = \kappa^{N_g}(Y)O^{N_g}(Y, f).$$

6.3. **Semisimple descent.** We recall the semisimple descent of orbital integrals, which is our main tool in proving Theorem 6.3. This is a very general procedure, so we temporarily consider in this subsection a reductive group G acting on an affine variety X and $x \in X(F)$ is a semisimple point (i.e. the G-orbit of x is Zariski closed, or equivalently $G(F)x$ is closed in $X(F)$ in the analytic topology). We will need the notion of an analytic Luna slice, cf. [AG09]. The analytic Luna slice at x is denoted by (U, p, ψ, M, N_x), where

- U is an $G(F)$-invariant analytic neighbourhood of x in $X(F)$.
- p is an $G(F)$-equivariant analytic retraction $p : U \to G(F)x$ and $M = p^{-1}(x)$.
- ψ is an $G(F)_x$-equivariant analytic embedding $M \to N_x$ with saturated image and $\psi(x) = 0$.

Here saturated means that $M = \psi^{-1}(\psi(M))$. Note that since p is $G(F)$-equivariant, if $y \in M$ and $gy = y$ where $g \in G(F)$, then $gx = x$, i.e. $G(F)_y$ is a subgroup of $G(F)_x$.

For semisimple $x \in X(F)$, we have an étale Luna slice Z and strongly etale morphisms $\iota : Z \to N_x$ and $\phi : G \times_{G_x} Z \to X$. We can construct an analytic slice (U, p, ψ, M, N_x) from it, cf. [AG09, Corollary A.2.4]. Let $\pi_Z : Z \to Z//G_x$ be the categorical quotient. By definition, the morphisms $Z//G_x \to X//G$ and $Z//G_x \to N_x//G_x$ are both étale. Therefore we may choose a sufficiently small analytic neighbourhood Z' of $\pi_Z(x)$ in $(Z//G_x)(F)$, so that the above two morphisms are (analytic) isomorphisms from Z' to its image. Let M be the inverse image of Z' under the natural map $Z(F) \to (Z//G_x)(F)$. Let $\psi = \iota|_M$. Let U' be the inverse image of Z' in $(G \times_{G_x} Z)(F)$. Let $p' : U' \to (G//G_x)(F)$ be the natural $G(F)$-invariant map. Let $U'' = U' \cap p'^{-1}(G(F)//G(F)_x)$ and $U = \phi(U'') \subset X(F)$. Note that $\phi|_{U''}$ is an analytic isomorphism from U'' to U. Let $p = p' \circ (\phi|_{U''})^{-1}$. Then (U, p, ψ, M, N_x) is the desired analytic Luna slice at x.

The following is [Zha14, Proposition 3.11]. It describes the relation between the orbital integrals on X near the semisimple point x and the orbital integrals on the sliced representation at x.

Proposition 6.7. *Let χ be a character of $G(F)$. There is an H_x-invariant neighbourhood $\mathcal{V} \subset M$ of x such that the following holds.*

(1) *For any $f \in \mathcal{S}(X(F))$ there is an $f_x \in \mathcal{S}(N_x)$ such that if $y \in \mathcal{V}$, $z = \psi(y)$ and χ is trivial on $H(F)_y$, we have*

$$(6.2) \qquad \int_{G(F)/G_y(F)} f(gy)\chi(g)\mathrm{d}g = \int_{G_x(F)/G_y(F)} f_x(gz)\chi(g)\mathrm{d}g.$$

(2) *Conversely for any $f_x \in \mathcal{S}(N_x)$ there is an $f \in \mathcal{S}(X(F))$ such that the equality (6.2) holds if $y \in \mathcal{V}$, $z = \psi(y)$ and χ is trivial on $H(F)_y$.*

6.4. **Proof of Theorem 6.3.** Let us retain the setup of Theorem 6.3. Let $s' = s'(\alpha, n_1, n_2, n_3) \in S'(F)$ and $g = g(\beta, n_1, n_2, n_3) \in G(F)$ be semisimple elements.

Lemma 6.8. *If g does not match any semisimple element in $S'(F)$, then there is an $(H \times H)(F)$-invariant neighbourhood U of g such that $U \cap S(F)_{\mathrm{reg},0} = \emptyset$. If s' does not match any semisimple element in $G(F)$, then there is an $H'(F)$-invariant neighbourhood U' of s' such that $U' \cap S'(F)_{\mathrm{reg},0} = \emptyset$.*

Proof. That g does not match any semisimple element in $S'(F)$ is equivalent to

$$1 - \epsilon\beta\bar{\beta} \notin N\,\mathrm{GL}_{n_1}(E).$$

Any regular semisimple element in a neighbourhood of g is in the $(H \times H)(F)$ orbit of an element $y = (y_1, y_2, y_3) \in U$. We can choose the neighbourhood to be small enough such that if $y_1\theta(y_1)^{-1} = \begin{pmatrix} A & \epsilon B \\ \bar{B} & \bar{A} \end{pmatrix}$ then $\frac{1}{2}(A + 1)$ is in a small neighbourhood of $(1 - \epsilon\beta\bar{\beta})^{-1}$ in $\mathrm{GL}_{n_1}(E)$. When this neighbourhood is small enough, no regular semisimple elements (in the usual sense) in it is a norm (see below for an explanation). This shows that $y = (y_1, y_2, y_3)$ does not match any element in $S'(F)$.

The case of s' can be proved by the same argument.

It remains to explain that if $\gamma \in \mathrm{GL}_{n_1}(F)$ is semisimple in the usual sense and $\gamma \notin N\,\mathrm{GL}_{n_1}(E)$, then there is a neighbourhood of γ such that any regular semisimple δ in the neighbourhood is not a norm. This can be seen as follows. Suppose that there is a sequence of regular semisimple elements δ_k, such that δ_k converges to γ and $\delta_k = Ng_k$ for some $g_k \in \mathrm{GL}_{n_1}(E)$. As δ_k's are all conjugate to an element in $\mathrm{GL}_{n_1}(F)$, we may assume that $\delta_k \in \mathrm{GL}_{n_1}(F)$. The conjugation action map $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_1}(F)_\gamma \to \mathrm{GL}_{n_1}(F)$, $(x, y) \mapsto xyx^{-1}$, is a fibration in a neighbourhood of γ. Therefore we may further assume that $\delta_k \in \mathrm{GL}_{n_1}(F)_\gamma$. Moreover as there are only finitely many maximal tori in $\mathrm{GL}_{n_1}(F)_\gamma$ up to conjugation, we may assume, by taking a subsequence, that there is a maximal torus T of $\mathrm{GL}_{n_1}(F)_\gamma$, such that $\delta_k \in T(F)$. As δ_k are all regular semisimple, we

conclude that $g_k \in T(E)$. Since the norm map $T(E) \to T(F)$ is locally a fibration, we may assume, again by taking a subsequence, that g_k is convergent to an element g. Then $\gamma = Ng$. □

From now on let us assume that g and s' match. This in particular implies that $-(1-\alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}$ and $\epsilon\beta\overline{\beta}$ have the same characteristic polynomial. We may and will assume that they in fact equal. Put

$$s'_1 = \begin{pmatrix} \alpha & 1_{n_1} \\ 1_{n_1} - \alpha\overline{\alpha} & -\overline{\alpha} \end{pmatrix}, \quad s'_2 = \begin{pmatrix} & 1_{n_2} \\ 1_{n_2} & \end{pmatrix}, \quad s'_3 = 1_{2n_3},$$

and

$$g_1 = \begin{pmatrix} 1_{n_1} & \epsilon\beta \\ \overline{\beta} & 1_{n_1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} & 1_{n_2} \\ 1_{n_2} & \end{pmatrix}, \quad g_3 = 1_{2n_3}.$$

The stabilizer of s (resp. s') has the form $H_1 \times H_2 \times H_3$ (resp. $H'_1 \times H'_2 \times H'_3$), and the sliced representation N_s (resp. $N_{s'}$) has the form $V_1 \times V_2 \times V_3$ (resp. $V'_1 \times V'_2 \times V'_3$). We have $V_i//H_i \simeq V'_i//H'_i$. By the explicit construction in Subsection 5.2, the étale Luna slice at g (resp. s') takes the form $Z = Z_1 \times Z_2 \times Z_3$ (resp. $Z' = Z'_1 \times Z'_2 \times Z'_3$). The analytic slice at g (resp. s') is denoted by (U, p, ψ, M, N_g) (resp. $(U', p', \psi', M', N_{s'})$). According to the construction of the analytic slice recalled in Subsection 6.3, M (resp. M') takes the form $M_1 \times M_2 \times M_3$ (resp. $M'_1 \times M'_2 \times M'_3$) where $M_i \subset Z_i$ (resp. $M'_i \subset Z'_i$), where $\psi'(M_i)$ (resp. $\psi'(M'_i)$) is a saturated open subset of V_i (resp. V'_i). We may assume that the image of M_i (resp. M'_i) in $V_i//H_i = V'_i//H'_i$ are identified.

Let us assume that $xs' = (x_1 s'_1, x_2 s'_2, x_3 s'_3) \in M'$ be a regular semisimple element, $x_i s'_i \in M_i$, and

$$\psi'(xs') = \left(\begin{pmatrix} & X_1 \\ -\overline{X_1}(1-\alpha\overline{\alpha}) & \end{pmatrix}, \begin{pmatrix} & X_2 \\ -\overline{X_2} & \end{pmatrix}, \begin{pmatrix} & X_3 \\ X_4 & \end{pmatrix} \right),$$

where $X_1 \in V'_1$, $X_2 \in V'_2$ and $(X_3, X_4) \in V'_3$. Write $X = (X_1, X_2, (X_3, X_4)) \in N_{s'}$. Let $yg = (y_1 g_1, y_2 g_2, y_3 g_3) \in M$ be a regular semisimple elements, $y_i g_i \in M_i$, and

$$\psi(yg) = \left(\begin{pmatrix} & \epsilon Y_1 \\ \overline{Y_1} & \end{pmatrix}, \begin{pmatrix} & \epsilon Y_2 \\ \overline{Y_2} & \end{pmatrix}, \begin{pmatrix} & Y_3 \\ \overline{Y_3} & \end{pmatrix} \right),$$

where $Y_i \in V_i$. Write $Y = (Y_1, Y_2, Y_3) \in N_g$.

The next lemma connects the notion of matching of xs' and yg with the matching of X and Y in the sliced representation. We use $a \sim b$ to indicate that a and b have the same characteristic polynomial. We also note that $X_1\overline{\alpha}$ and $\alpha\overline{\alpha}$ commute, and hence it makes sense to evaluate a power series with coefficients being rational functions of $\alpha\overline{\alpha}$ at $X_1\overline{\alpha}$. Similarly it makes sense to evaluate a power series with coefficients being rational functions of $\epsilon\beta\overline{\beta}$ at $Y_1\overline{\beta}$.

Lemma 6.9. *We can find an H'_i-invariant neighbourhood $\mathcal{V}'_i$ of s'_i in M'_i and a neighbourhood $\mathcal{V}_i$ of g_i in M_i, $i = 1, 2, 3$, and a power series ξ_1 with coefficients being rational functions of $\alpha\overline{\alpha}$, and the leading term*

$$\xi_1(t) = t(1 - \alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1} + \cdots,$$

such that if xs' and yg match, and $x_i s'_i \in \mathcal{V}'_i$, $y_i g_i \in \mathcal{V}_i$, $i = 1, 2, 3$, then

$$\xi_1(X_1\overline{\alpha}) \sim Y_1\overline{\beta}, \quad X_2\overline{X_2} \sim -\epsilon Y_2\overline{Y_2}, \quad X_3\overline{X_3} \sim \epsilon Y_3\overline{Y_3}.$$

Proof. We always assume that x and y lie in a small neighbourhoods of 1, or equivalently X_i's and Y_i's are sufficiently close to 0. Elementary but tedious calculations give

$$x_1 s'_1 = \begin{pmatrix} A & B \\ -\overline{B}(1 - \alpha\overline{\alpha}) & \overline{A} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 - \alpha\overline{\alpha} & -\overline{\alpha} \end{pmatrix}$$

where

$$A = (1 + X_1\overline{X_1}(1 - \alpha\overline{\alpha}))^{-1}(1 - X_1\overline{X_1}(1 - \alpha\overline{\alpha})), \quad B = -2(1 + X_1\overline{X_1}(1 - \alpha\overline{\alpha}))^{-1}X_1$$

and

$$x_2 s'_2 = \begin{pmatrix} -2(1 + X_2\overline{X_2})^{-1}X_2 & (1 + X_2\overline{X_2})^{-1}(1 - X_2\overline{X_2}) \\ (1 + X_2\overline{X_2})^{-1}(1 - X_2\overline{X_2}) & 2(1 + X_2\overline{X_2})^{-1}X_2 \end{pmatrix},$$

and

$$x_3 s'_3 = \begin{pmatrix} (1 - X_3 X_4)^{-1}(1 + X_3 X_4) & -2(1 - X_3 X_4)^{-1}X_3 \\ -2(1 - X_3 X_4)^{-1}X_4 & (1 - X_3 X_4)^{-1}(1 + X_3 X_4) \end{pmatrix}.$$

Similarly we have

$$y_1 g_1 = \begin{pmatrix} (1 - \epsilon Y_1\overline{Y_1})^{-1}(1 + \epsilon Y_1\overline{Y_1}) & -2\epsilon(1 - \epsilon Y_1\overline{Y_1})^{-1}Y_1 \\ -2\overline{Y_1}(1 - \epsilon Y_1\overline{Y_1})^{-1} & (1 - \epsilon\overline{Y_1}Y_1)^{-1}(1 + \epsilon\overline{Y_1}Y_1) \end{pmatrix} \begin{pmatrix} 1 & \epsilon\beta \\ \overline{\beta} & 1 \end{pmatrix}$$

and

$$y_2 g_2 = \begin{pmatrix} -2\epsilon(1 - \epsilon Y_2\overline{Y_2})^{-1}Y_2 & \epsilon(1 - \epsilon Y_2\overline{Y_2})^{-1}(1 + \epsilon Y_2\overline{Y_2}) \\ (1 - \epsilon\overline{Y_2}Y_2)^{-1}(1 + \epsilon\overline{Y_2}Y_2) & -2\epsilon(1 - \epsilon\overline{Y_2}Y_2)^{-1}\overline{Y_2} \end{pmatrix}$$

and

$$y_3 g_3 = \begin{pmatrix} (1 - \epsilon Y_3\overline{Y_3})^{-1}(1 + \epsilon Y_3\overline{Y_3}) & -2\epsilon(1 - \epsilon Y_3\overline{Y_3})^{-1}Y_3 \\ -2\overline{Y_3}(1 - \epsilon Y_3\overline{Y_3})^{-1} & (1 - \epsilon\overline{Y_3}Y_3)^{-1}(1 + \epsilon\overline{Y_3}Y_3) \end{pmatrix}$$

From these expression it is straightforward to check that if xs' and yg match, and when X_2, X_3, X_4 and Y_2, Y_3 are close to 0, then

$$X_2\overline{X_2} \sim -\epsilon Y_2\overline{Y_2}, \quad X_3 X_4 \sim \epsilon Y_3\overline{Y_3}.$$

It remains to treat X_1 and Y_1. Write

$$x_1 s'_1 = \begin{pmatrix} A'_1 & * \\ * & * \end{pmatrix}.$$

A little computation gives $2A_1\overline{A_1} - 1$ is a rational function in $X_1\overline{\alpha}$ and $\alpha\overline{\alpha}$. Its power series expansion (in the variable $t = X_1\overline{\alpha}$) is $\xi'_1(X_1\overline{\alpha})$ where ξ'_1 with coefficients in rational functions of $\alpha\overline{\alpha}$ and the first few terms are

$$2\alpha\overline{\alpha} - 1 - 8t(1 - \alpha\overline{\alpha}) + \cdots$$

The power series is convergent when $X_1\overline{\alpha}$ is sufficiently close to zero.

Write

$$y_1 g_1 \theta(y_1 g_1)^{-1} = \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix}.$$

A little computation gives A_1 is a rational function in $\epsilon\beta\overline{\beta}$ and $Y_1\overline{\beta}$. Its power series expansion (in the variable $t = Y_1\overline{\beta}$ is $\widetilde{\xi_1}(Y_1\overline{\beta})$ where ξ_1 is a power series whose coefficients are rational functions in $\epsilon\beta\overline{\beta}$ and the first few terms are

$$(1 + \epsilon\beta\overline{\beta})(1 - \epsilon\beta\overline{\beta})^{-1} - 8t(1 - \epsilon\beta\overline{\beta})^{-1} + \cdots$$

The power series is convergent when $Y_1\overline{\beta}$ is sufficiently close to zero.

That xs' and yg match implies that $\xi_1'(X_1\overline{\alpha}) \sim \widetilde{\xi_1}(Y_1\overline{\beta})$. By assumption we have $-(1 - \alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1} = \epsilon\beta\overline{\beta}$ and $\det(1 - \alpha\overline{\alpha}) \neq 0$. This implies that the constant terms of ξ_1' and $\widetilde{\xi_1}$ equal. So there is a power series ξ_1 with coefficients in $\alpha\overline{\alpha}$ of the form

$$t(1 - \alpha\overline{\alpha})^{-1}(\alpha\overline{\alpha}) + \cdots,$$

such that $\xi_1(X_1\overline{\alpha}) \sim Y_1\overline{\beta}$. $\qquad\square$

Proof of Theorem 6.3. Let $f' \in \mathcal{S}(S'(F))_0$. We prove that there is an $f \in \mathcal{S}(G(F))_0$ that matches it. The other direction is similar. By lemma 6.4, it is enough to prove such an f exists locally near every semisimple point s'. Moreover precisely, we will prove that for any semisimple point $s' \in S'(F)$, there is an $f \in \mathcal{S}(G(F))_0$ such that

(6.3) $$\kappa^{S'}(xs')O^{S'}(xs', f') = \kappa^G(yg)O^G(yg, f)$$

for all matching regular semisimple $xs' \in S'(F)$ and $yg \in G(F)$ when x and y are sufficiently close to 1.

We use semisimple descent at the point s'. First according to the computation in the proof of Lemma 6.9 when $X = \psi'(xs') \in N_{s'}$ is sufficiently close to 0, we have that $\kappa^{S'}(xs')$ is a constant multiple of $\kappa^{N_{s'}}(X)$.

Let us now apply Proposition 6.7 to the orbital integrals $O^{S'}(xs', f')$. There is a small enough H_i'-invariant neighbourhood $\mathcal{V}_i' \subset M_i'$ of 0, and a function $f_{s'}' \in \mathcal{S}(\psi'(\mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3'))$ such that for all $xs' \in \mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3'$, we have

$$\kappa^{S'}(xs')O^{S'}(xs', f') = \kappa^{N_{s'}}(X)O^{N_{s'}}(X, f_{s'}').$$

We put $\mathcal{U}_i' = \psi_i'(\mathcal{V}_i')$, $i = 1, 2, 3$, $\mathcal{V}' = \mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3'$, $\mathcal{U}' = \psi'(\mathcal{V}') = \mathcal{U}_1' \times \mathcal{U}_2' \times \mathcal{U}_3'$.

For $i = 1, 2, 3$, by shrinking $\mathcal{V}_i$, we may assume that the neighbourhood $\mathcal{V}_i'$ such that Lemma 6.9 holds. We can find a power series ξ_1 with coefficients in rational functions of $\alpha\overline{\alpha}$ such that if xs' and yg match and $x_i s_i' \in \mathcal{V}_i'$, $i = 1, 2, 3$, then

$$\xi_1(X_1\overline{\alpha}) \sim Y_1\overline{\beta}, \quad X_2\overline{X_2} \sim -\epsilon Y_2\overline{Y_2}, \quad X_3 X_4 \sim \epsilon Y_3\overline{Y_3}.$$

Shrinking $\mathcal{V}_1'$ further if necessary, we may assume that $X_1 \mapsto \xi_1(X_1\overline{\alpha})\overline{\alpha}^{-1}$ defines a bijection from $\mathcal{U}_1$ to its image in V_1'. Let $\xi : \mathcal{U}' \to N_{s'}$ be the map

$$\xi(X_1, X_2, (X_3, X_4)) = (\xi_1(X_1\overline{\alpha})\overline{\alpha}^{-1}, X_2, (X_3, X_4)).$$

Note that this map is $H_{s'}'$-equivariant. Then $\xi(\mathcal{U}')$ is a neighbourhood of $0 \in N_{s'}$ and ξ is a bijection from $\mathcal{U}'$ to its image. Moreover $\kappa^{N_{s'}}(X) = a\kappa^{N_{s'}}(\xi(X))$ if $X \in \mathcal{U}'$, where a is a nonzero constant.

Define a function $\widetilde{f_{s'}'}$

$$\widetilde{f_{s'}'}(X) = \begin{cases} af_{s'}'(\xi^{-1}(X)), & X \in \xi(\mathcal{U}'), \\ 0, & X \notin \xi(\mathcal{U}'). \end{cases}$$

Then we have

$$\kappa^{N_{s'}}(X)O^{N_{s'}}(X, f_{s'}') = \kappa^{N_{s'}}(\xi'(X))O^{N_{s'}}(\xi'(X), \widetilde{f_{s'}'}),$$

for all $X \in \xi(\mathcal{U}')$. We view $f_{s'}'$ as an function on $N_{s'}$ via extension by zero. Since $f' \in \mathcal{S}(S'(F))_0$, we conclude that $\widetilde{f_{s'}'} \in \mathcal{S}(N_{s'})_0$.

By Proposition 6.6 there is an $f_g \in \mathcal{S}(N_g)_0$ that matches $\widetilde{f_{s'}'}$, i.e.

$$\kappa^{N_{s'}}(\xi(X))O^{N_{s'}}(\xi(X), \widetilde{f_{s'}'}) = \kappa^{N_g}(Y)O^{N_g}(Y, f_g),$$

for all regular semisimple matching $\xi(X) \in \xi(\mathcal{U}')$ and $Y \in N_g$.

As in the case of S', when Y is sufficiently close to 0 in N_g, we have $\kappa^G(yg)$ equals a constant multiple of $\kappa^{N_g}(Y)$. Applying Proposition 6.7 in the converse direction, we conclude that there is an H_s-invariant neighbourhood $\mathcal{V}$ of g in N_g, and an $f \in \mathcal{S}(G(F))$ such that

$$\kappa^{N_g}(Y)O^{N_s}(Y, f_g) = \kappa^G(yg)O^S(yg, f)$$

for all $yg \in \mathcal{V}$. Since $f_g \in \mathcal{S}(N_g)_0$ we conclude that $f \in \mathcal{S}(G(F))_0$. Shrinking $\mathcal{V}$ if necessary, we may assume that if $yg \in \mathcal{V}$ and $xs' \in S'(F)$ match then $xs' \in \mathcal{V}'$. Then we conclude a chain of equalities

$$\kappa^G(yg)O^S(yg, f) = \kappa^{N_g}(Y)O^{N_s}(Y, f_g) = \kappa^{N_{s'}}(\xi(X))O^{N_{s'}}(\xi(X), \widetilde{f_{s'}'})$$
$$= \kappa^{N_{s'}}(X)O^{N_{s'}}(X, f_{s'}') = \kappa^{S'}(xs')O^{S'}(xs', f')$$

when $xs' \in \mathcal{V}'$ and $yg \in \mathcal{V}$ match. This proves (6.3) and thus Theorem 6.3. $\qquad \square$

7. The fundamental lemma

7.1. **The fundamental lemma.** Assume that E/F is unramified with odd residue characteristic. Let $\widetilde{\eta}(x) = (-1)^{\mathrm{val}_E(x)}$ be the unique unramified character that extends η to $E^\times$. Let $\tau \in \mathfrak{o}_E^\times$ be a purely imaginary element which is used in the definition of the transfer factor $\kappa^{S'}$ (cf. (6.1)).

Theorem 7.1. *Let $s' = s'(\alpha) \in S'(F)$ and $g = g(\beta) \in G(F)$ be matching regular semisimple elements. Then*

(7.1) $$\kappa^{S'}(s')O^{S'}(s', \mathbf{1}_{S'(\mathfrak{o}_F)}) = \kappa^G(g)O^G(g, \mathbf{1}_{G(\mathfrak{o}_F)}).$$

It is straightforward to see that this theorem implies Theorem 2.2. The rest of this section is devoted to prove this theorem. We distinguish two cases depending on $\overline{\alpha}\alpha$ being elliptic or not. The elliptic case will be deduced from the base change fundamental lemma and the calculations in [Guo96]. The nonelliptic case will be treated using the parabolic descent and induction on n.

Before we prove the theorem, let us first explain that the validity of (7.1) is independent of the character χ. In fact, the left hand side equals

$$\chi(-\overline{\alpha})\int \mathbf{1}_{S'(\mathfrak{o}_F)}\left(\begin{pmatrix}h_1^{-1} & \\ & h_2^{-1}\end{pmatrix}\begin{pmatrix}\alpha & 1 \\ 1-\alpha\overline{\alpha} & -\overline{\alpha}\end{pmatrix}\begin{pmatrix}\overline{h_1} & \\ & \overline{h_2}\end{pmatrix}\right)\chi(\overline{h_2}h_2^{-1})\widetilde{\eta}(h_1h_2)\mathrm{d}h_1\mathrm{d}h_2.$$

Since $\det \overline{h_2}h_2^{-1}\in\mathfrak{o}_E^\times$ and χ is unramified, the above expression equals

$$\chi(-\overline{\alpha})\int \mathbf{1}_{S'(\mathfrak{o}_F)}\left(\begin{pmatrix}h_1^{-1} & \\ & h_2^{-1}\end{pmatrix}\begin{pmatrix}\alpha & 1 \\ 1-\alpha\overline{\alpha} & -\overline{\alpha}\end{pmatrix}\begin{pmatrix}\overline{h_1} & \\ & \overline{h_2}\end{pmatrix}\right)\widetilde{\eta}(h_1h_2)\mathrm{d}h_1\mathrm{d}h_2.$$

The right hand side of (7.1) equals

$$\int \mathbf{1}_{G(\mathfrak{o}_F)}\left(\begin{pmatrix}g^{-1} & \\ & \overline{g}^{-1}\end{pmatrix}\begin{pmatrix}1 & \beta \\ \overline{\beta} & 1\end{pmatrix}\begin{pmatrix}h & \\ & \overline{h}\end{pmatrix}\right)\chi(g^{-1}h)\mathrm{d}g\mathrm{d}h.$$

Taking determinant we see that the integrand is nonzero only when

$$\det(g\overline{g})^{-1}\det(1-\beta\overline{\beta})\det(h\overline{h})\in\mathfrak{o}_F^\times.$$

Recall that $s'(\alpha)$ and $g(\beta)$ match if $-(1-\alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}$ and $\beta\overline{\beta}$ have the same characteristic polynomial, which implies that $(1-\beta\overline{\beta})^{-1}$ and $\alpha\overline{\alpha}$ have the same characteristic polynomial. As χ is unramified we obtain $\chi(g^{-1}h)=\chi(\alpha)=\chi(-\overline{\alpha})$. Thus the right hand side of (7.1) equals

$$\chi(-\overline{\alpha})\int \mathbf{1}_{G(\mathfrak{o}_F)}\left(\begin{pmatrix}g^{-1} & \\ & \overline{g}^{-1}\end{pmatrix}\begin{pmatrix}1 & \beta \\ \overline{\beta} & 1\end{pmatrix}\begin{pmatrix}h & \\ & \overline{h}\end{pmatrix}\right)\mathrm{d}g\mathrm{d}h.$$

Thus from now on we assume that χ is trivial. Under this assumption the transfer factors on both sides of (7.1) are trivial.

7.2. The elliptic case.
Put $r=-(1-\alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}$ and

$$x_r=|\det(1-r)|=|\det\alpha\overline{\alpha}|^{-1}, \quad y_r=|\det r|=|\det(1-\alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}|.$$

In this subsection we always assume that $\alpha\overline{\alpha}$ and hence r are elliptic regular semisimple in $\mathrm{GL}_n(F)$ in the usual sense. Note that this in particular covers the base case of the induction $n=1$.

Make a change of variable $h_1\mapsto \overline{h_2}h_1$ on the left hand side of (7.1) we obtain

$$(7.2)\qquad \int \mathbf{1}_{S'(\mathfrak{o}_F)}\left(\begin{pmatrix}h_1^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}\overline{h_2}^{-1}\alpha h_2 & 1 \\ h_2^{-1}(1-\overline{\alpha}\alpha)h_2 & -h_2^{-1}\overline{\alpha}\overline{h_2}\end{pmatrix}\begin{pmatrix}\overline{h_1} & \\ & 1\end{pmatrix}\right)\widetilde{\eta}(\det h_1)\mathrm{d}h_1\mathrm{d}h_2.$$

Here the integration is over $h_2\in\mathrm{GL}_n(E)_{\alpha,\text{twisted}}\backslash\mathrm{GL}_n(E)$ and $h_1\in\mathrm{GL}_n(E)$.

The first observation is that if $x_r<1$, i.e. $|\det\alpha|>1$, then $h_2^{-1}\alpha\overline{h_2}\notin M_n(\mathfrak{o}_E)$ for any h_2 and therefore the integral vanishes. Moreover under this condition, the right hand side of (7.1) also vanishes by [Guo96, (3.9)]. Thus Theorem 7.1 holds in this case.

Assume from now on that $x_r \geq 1$. By [Guo96, Lemma 3.4], either $y_r \leq x_r = 1$ or $x_r = y_r > 1$. We distinguish two cases.

(1) $y_r \leq x_r = 1$, i.e. $\det \alpha \in \mathfrak{o}_E^\times$. Then $y_r = |\det(1 - \overline{\alpha}\alpha)|$.

We need to make use of a lemma of Kottwitz's, cf. [Kot80, Lemma 8.8], which we recall in a special case for readers' convenience. The lemma says that if $\gamma \in \mathrm{GL}_n(E)$ is conjugate to an element in $\mathrm{GL}_n(\mathfrak{o}_E)$ and is regular semisimple, then we can find an element $\delta \in \mathrm{GL}_n(E)$ such that $\gamma = \delta\overline{\delta} = \overline{\delta}\delta$ and $x^{-1}\gamma x \in \mathrm{GL}_n(\mathfrak{o}_E)$ implies $x^{-1}\delta x \in \mathrm{GL}_n(\mathfrak{o}_E)$. Moreover if δ satisfies the above property and $h \in \mathrm{GL}_n(E)$, then $h^{-1}\overline{\delta}h \in \mathrm{GL}_n(\mathfrak{o}_E)$ if and only if $h^{-1}\gamma h \in \mathrm{GL}_n(\mathfrak{o}_E)$ and $h \in \mathrm{GL}_n(F)\,\mathrm{GL}_n(\mathfrak{o}_E)$.

Let us come back to our setup. If $\alpha\overline{\alpha}$ is not conjugate to an element in $\mathrm{GL}_n(\mathfrak{o}_F)$, by considering the lower right block of the matrix, we see that (7.2) equals zero. Let us assume that $\alpha\overline{\alpha}$ is conjugate to an element in $\mathrm{GL}_n(\mathfrak{o}_F)$. We first apply this lemma to $\gamma = \alpha\overline{\alpha}$. We may further assume that $\alpha = \delta$ has the property described in the lemma of Kottwitz's. We then conclude that $h_2 \in \mathrm{GL}_n(F)\,\mathrm{GL}(\mathfrak{o}_E)$. Thus we may replace the outer integral in (7.2) by $h_2 \in \mathrm{GL}_n(F)_\alpha \backslash \mathrm{GL}_n(F)$. Here though α might not be in $\mathrm{GL}_n(F)$ but in $\mathrm{GL}_n(E)$, the group $\mathrm{GL}_n(F)_\alpha$ stands for all $h \in \mathrm{GL}_n(F)$ that commutes with α. We note that $\mathrm{GL}_n(F)_\alpha = \mathrm{GL}_n(F)_{\alpha\overline{\alpha}}$. Indeed as explained in [AC89, Chapter 1, Proof of Lemma 1.1], as algebraic groups over F, the group $\mathrm{GL}_n(E)_{\alpha,\mathrm{twisted}}$ is an inner form of $\mathrm{GL}_n(F)_{\alpha\overline{\alpha}}$. Since $\alpha\overline{\alpha}$ is regular semisimple, both are tori over F and hence are canonically isomorphic, which implies that $\mathrm{GL}_n(E)_{\alpha,\mathrm{twisted}} = \mathrm{GL}_n(F)_{\alpha\overline{\alpha}}$. This in particular implies that if $h \in \mathrm{GL}_n(E)_{\alpha,\mathrm{twisted}}$ then $h \in \mathrm{GL}_n(F)$ and thus $\mathrm{GL}_n(F)_\alpha = \mathrm{GL}_n(E)_{\alpha,\mathrm{twisted}} = \mathrm{GL}_n(F)_{\alpha\overline{\alpha}}$. Therefore we may replace the outer integral by $h_2 \in \mathrm{GL}_n(F)_{\alpha\overline{\alpha}} \backslash \mathrm{GL}_n(F)$.

Now $h_2 \in \mathrm{GL}_n(F)$ and we apply the lemma again to $\gamma = h_2^{-1}\alpha\overline{\alpha}h_2$, and $\delta = h_2^{-1}\overline{\alpha}h_2$. Clearly this γ and δ again satisfy the conditions in the lemma of Kottwitz's. By consider the upper left corner of the matrix in the integrand, we conclude that $h_1 \in \mathrm{GL}_n(F)\,\mathrm{GL}(\mathfrak{o}_E)$ and we may replace the outer integral in (7.2) by $h_1 \in \mathrm{GL}_n(F)$.

The domain of integral in (7.2) is thus equivalent to the four conditions

$$h_1^{-1}h_2^{-1}\alpha h_2 h_1,\ h_2^{-1}\alpha h_2 \in \mathrm{GL}_n(\mathfrak{o}_E), \quad h_1^{-1},\ h_2^{-1}(1 - \alpha\overline{\alpha})h_2 h_1 \in M_n(\mathfrak{o}_F).$$

As α satisfies the conditions in Kottwitz's lemma, the first two conditions are implied by the other two. To see this, observe that h_1^{-1} and $h_2^{-1}(1 - \overline{\alpha}\alpha)h_2 h_1$ being in $M_n(\mathfrak{o}_F)$ implies both

$$h_2^{-1}(1 - \alpha\overline{\alpha})h_2, \quad h_1^{-1}h_2^{-1}(1 - \alpha\overline{\alpha})h_2 h_1$$

are in $M_n(\mathfrak{o}_F)$. As we have assume that $\det \alpha \in \mathfrak{o}_E^\times$, we conclude that

$$h_2^{-1}\alpha\overline{\alpha}h_2, \quad h_1^{-1}h_2^{-1}\alpha\overline{\alpha}h_2 h_1$$

are in $\mathrm{GL}_n(\mathfrak{o}_F)$. That α satisfies the conditions in the lemma of Kottwitz's implies

$$h_1^{-1}h_2^{-1}\alpha h_2 h_1, \quad h_2^{-1}\alpha h_2$$

are both in $\mathrm{GL}_n(\mathfrak{o}_E)$.

The integral (7.2) thus simplifies to

$$\int \mathbf{1}_{M_n(\mathfrak{o}_F) \times M_n(\mathfrak{o}_F)}(h_2^{-1}(1 - \alpha\overline{\alpha})(\alpha\overline{\alpha})^{-1}h_2 h_1, h_1^{-1})\eta(\det h_1)\mathrm{d}h_1\mathrm{d}h_2,$$

where the domain of integration is $h_1 \in \mathrm{GL}_n(F)$ and $h_2 \in \mathrm{GL}_n(F)_{\alpha\overline{\alpha}} \backslash \mathrm{GL}_n(F)$.

In [Guo96, Lemma 3.5], a Hecke function Ψ_r on $\mathrm{GL}_n(F)$ is defined. By the calculation in [Guo96, p. 137], under the assumption that $y_r \leq x_r = 1$, this function equals

$$g \mapsto \int_{\mathrm{GL}_n(F)} \mathbf{1}_{\{X,Y \in M_n(\mathfrak{o}_F), \ |\det XY|=y_r\}}(gh_1, h_1^{-1})\eta(h_1)\mathrm{d}h_1.$$

Therefore (7.2) equals

(7.3)
$$\int_{\mathrm{GL}_n(F)_{\alpha\overline{\alpha}} \backslash \mathrm{GL}_n(F)} \Psi_r(h_2^{-1}rh_2)\mathrm{d}h_2.$$

The right hand side of (7.1) is exactly the orbital integral appearing in [Guo96]. For any (twisted) elliptic regular semisimple $\beta \in \mathrm{GL}_n(E)$, a Heck function Φ_β on $\mathrm{GL}_n(E)$ is defined in [Guo96, Lemma 3.6]. By the calculation in [Guo96, p. 139], under the assumption $y_r \leq x_r = 1$, Φ_β is the characteristic function of

$$\{X \in M_n(\mathfrak{o}_E), |\det X| = y_r\}.$$

By [Guo96, Lemma 3.6] the right hand side of (7.1) equals

(7.4)
$$\int_{\mathrm{GL}_n(E)_{\beta,\mathrm{twisted}} \backslash \mathrm{GL}_n(E)} \Phi_\beta(h^{-1}\beta\overline{h})\mathrm{d}h.$$

If $\alpha\overline{\alpha}$ is not conjugate to an element in $\mathrm{GL}_n(\mathfrak{o}_F)$, then neither is $1 - \beta\overline{\beta}$. In this case the above integral of Φ_β equals zero so both sides of (7.1) equal zero. Now assume that $\alpha\overline{\alpha}$ is conjugate to an element in $\mathrm{GL}_n(\mathfrak{o}_F)$. Let $\mathcal{H}(\mathrm{GL}_n(E))$ and $\mathcal{H}(\mathrm{GL}_n(F))$ be the spherical Hecke algebra of $\mathrm{GL}_n(E)$ and $\mathrm{GL}_n(F)$ respectively, and

$$\mathrm{bc} : \mathcal{H}(\mathrm{GL}_n(E)) \to \mathcal{H}(\mathrm{GL}_n(F))$$

the usual base change map. By [Guo96, Corollary 3.8], we have $\Psi_r = \mathrm{bc}(\Phi_\beta)$. Thus the desired equality (7.1) is a consequence of the identities (7.3) (7.4), and the base change fundamental lemma [AC89, Chapter 1, Theorem 4.5].

(2) Let us now assume that $x_r = y_r > 1$, i.e. $|\det \alpha| < 1$ and hence $|\det(1 - \overline{\alpha}\alpha)| = 1$. The integrand of (7.2) is equivalent to the condition that

$$h_2^{-1}\overline{\alpha}\overline{h_2}, \quad h_1^{-1}\overline{h_2}^{-1}\alpha h_2\overline{h_1}, \quad h_2^{-1}(1 - \overline{\alpha}\alpha)h_2\overline{h_1}, \quad h_1^{-1}$$

are all in $M_n(\mathfrak{o}_E)$. Note that

$$|\det h_1^{-1}| \leq 1, \quad |\det h_2^{-1}(1 - \overline{\alpha}\alpha)h_2\overline{h_1}| \leq 1,$$

but

$$|\det(1 - \overline{\alpha}\alpha)| = 1.$$

It follows that $|\det h_1| = 1$ and hence $h_1 \in \mathrm{GL}_n(\mathfrak{o}_E)$. Moreover since $|\det \alpha| < 1$, we have $h_2^{-1}\overline{\alpha}\overline{h_2} \in M_n(\mathfrak{o}_E)$ implies $h_2^{-1}(1 - \overline{\alpha}\alpha)h_2 \in \mathrm{GL}_n(\mathfrak{o}_E)$. It follows that the integral (7.2) reduces to

$$(7.5) \qquad \int \mathbf{1}_{M_n(\mathfrak{o}_E)}(h_2^{-1}\alpha\overline{h_2})\mathrm{d}h_2.$$

By [Guo96, Lemma 3.6], and the calculation in [Guo96, p. 139-140], under the assumption $x_r = y_r > 1$, the right hand side of (7.1) equals

$$(7.6) \qquad \int \mathbf{1}_{\{X^{-1}\in M_n(\mathfrak{o}_E),\,|\det X|_E = x_r\}}(h_2^{-1}\beta\overline{h_2})\mathrm{d}h_2.$$

It remains to explain that the integrals (7.5) and (7.6) equal. First the condition $|\det X|_E = x_r$ in (7.6) is redundant as twisted conjugation does not change the absolute value of the determinant. As E/F is unramified, $-1 \in NE^\times$ and hence there is a $\delta \in \mathfrak{o}_E^\times$ such that $\delta\overline{\delta} = -1$. Since $\alpha\overline{\alpha}$ is elliptic and $|\det \alpha\overline{\alpha}| < 1$, we conclude the absolute values of its eigenvalues are all strictly less than one (in a fixed splitting field of F). Thus

$$(1 - \alpha\overline{\alpha})^{-\frac{1}{2}} = 1 - \binom{\frac{1}{2}}{1}\alpha\overline{\alpha} + \binom{\frac{1}{2}}{2}(\alpha\overline{\alpha})^2 + \cdots$$

is convergent and gives a well-defined element in $\mathrm{GL}_n(F)$. Put $\gamma = \delta(1 - \alpha\overline{\alpha})^{-\frac{1}{2}}$. By assumption $\alpha\overline{\alpha} \in \mathrm{GL}_n(F)$ and therefore γ commutes with α and thus $(\gamma\alpha\overline{\gamma\alpha})^{-1}$ and $\beta\overline{\beta}$ have the same characteristic polynomial. Replacing β by its twisted conjugate, we may assume that $\beta^{-1} = \gamma\alpha$. Therefore we need to explain

$$(7.7) \qquad (7.5) = \int \mathbf{1}_{M_n(\mathfrak{o}_E)}(h^{-1}\gamma\alpha\overline{h})\mathrm{d}h.$$

Assume that $h^{-1}\alpha\overline{h} \in M_n(\mathfrak{o}_E)$. Then $h^{-1}\alpha\overline{\alpha}h \in M_n(\mathfrak{o}_E)$ and the absolute values of all eigenvalues of it are strictly less than one. This implies

$$h^{-1}\gamma h = \delta\left(1 - \binom{\frac{1}{2}}{1}h^{-1}\alpha\overline{\alpha}h + \binom{\frac{1}{2}}{2}(h^{-1}\alpha\overline{\alpha}h)^2 + \cdots\right)$$

is convergent and is in $\mathrm{GL}_n(\mathfrak{o}_E)$ (note the only denominators in these binomial coefficients are powers of 2). Thus $h^{-1}\gamma\alpha\overline{h} \in M_n(\mathfrak{o}_E)$. Conversely if $h^{-1}\gamma\alpha\overline{h} \in M_n(\mathfrak{o}_E)$, then $h^{-1}(1 - \alpha\overline{\alpha})^{-1}\alpha\overline{\alpha}h \in M_n(\mathfrak{o}_E)$. Since $1 + h^{-1}(1 - \alpha\overline{\alpha})^{-1}\alpha\overline{\alpha}h = h^{-1}(1 - \alpha\overline{\alpha})^{-1}h$ and the absolute values of $\alpha\overline{\alpha}$ are all strictly less than one, we have $h^{-1}(1 - \alpha\overline{\alpha})h \in \mathrm{GL}_n(\mathfrak{o}_E)$ and hence $h^{-1}\alpha\overline{\alpha}h \in M_n(\mathfrak{o}_E)$. Then as before we conclude that $h^{-1}\gamma h \in \mathrm{GL}_n(\mathfrak{o}_E)$ and hence $h^{-1}\alpha\overline{h} \in M_n(\mathfrak{o}_E)$. This proves (7.7).

This finishes the proof of Theorem 7.1 when $\alpha\overline{\alpha}$ is elliptic.

7.3. Parabolic descent.

To handle the nonelliptic case, we make use of the parabolic descent of the orbital integrals. In this subsection, we deviate from the setup from Theorem 7.1 and consider $O^{G'}(x, f')$ where $f' \in \mathcal{S}(G'(F))$ and $x \in G'(F)$ is regular semisimple in general.

We fix integers n_1, n_2 with $n = n_1 + n_2$. Let $Q = LU$ be the parabolic subgroup of GL_{2n} of the form

$$
L = \begin{pmatrix} m_1^{(1)} & & m_1^{(2)} & \\ & m_2^{(1)} & & m_2^{(2)} \\ m_1^{(3)} & & m_1^{(4)} & \\ & m_2^{(3)} & & m_2^{(4)} \end{pmatrix}, \quad
U = \begin{pmatrix} 1 & u^{(1)} & & u^{(2)} \\ & 1 & & \\ & u^{(3)} & 1 & u^{(4)} \\ & & & 1 \end{pmatrix},
$$

where

$$
m_i = \begin{pmatrix} m_i^{(1)} & m_i^{(2)} \\ m_i^{(3)} & m_i^{(4)} \end{pmatrix} \in \mathrm{GL}_{2n_i}, \quad
\begin{pmatrix} u^{(1)} & u^{(2)} \\ u^{(3)} & u^{(4)} \end{pmatrix} \in M_{2n_1 \times 2n_2}.
$$

By definition

$$
O^{G'}(x, f') = (\chi\widetilde{\eta})^{-1}(x) \int_{(H' \times H'')_x \backslash (H' \times H'')} f'(h^{-1}xh'')\chi_{H'}(h)(\chi\eta)^{-1}(h'')\mathrm{d}h\mathrm{d}h''
$$

Suppose $x = (x_1, x_2) \in L(E)$ and $x_i \in \mathrm{GL}_{2n_i}(E)$, $i = 1, 2$. Assume that $x_i\overline{x_i}^{-1} = s'(\alpha_i)$ is regular semisimple in $S'_{n_i}(F)$. Here we add the subscript to indicate the size of the various groups and symmetric spaces. Let $\widetilde{r}_1, \cdots, \widetilde{r}_{n_1}$ and $\widetilde{s}_1, \cdots, \widetilde{s}_{n_2}$ be the eigenvalues of $\alpha_1\overline{\alpha_1}$ and $\alpha_2\overline{\alpha_2}$ respectively (in some fixed algebraic closure of F). Consider

$$
\prod_{1 \le i \le n_1, 1 \le j \le n_2} (\widetilde{r}_i - \widetilde{s}_j)^{-1}.
$$

Then one checks that it is an element in F. Let λ' be its absolute value.

Let $P = MN$ be the upper triangular parabolic subgroup of $\mathrm{GL}_n(E)$ corresponding to the partition $n = n_1 + n_2$. Here N is the unipotent radical, and M is the standard diagonal block Levi subgroup. Write $h = (h_1, h_2)$, $h_1, h_2 \in \mathrm{GL}_n(E)$. We make use of the Iwasawa decomposition

$$
h_i = u_i m_i k_i, \; i = 1, 2, \quad h'' = u''m''k''
$$

where $u_i \in N(E)$, $m_i \in M(E)$, $k_i \in \mathrm{GL}_n(\mathfrak{o}_E)$, $u' \in U(F)$, and $k' \in \mathrm{GL}_{2n}(\mathfrak{o}_F)$. Then $O^{G'}(x, f')$ equals

$$
(\chi\widetilde{\eta})^{-1}(x) \int f'_K(m^{-1}u^{-1}xu''m'')\delta_{P(E)}(m)^{-1}\delta_{Q(F)}(m'')^{-1}\chi_{H'}(m)(\chi\eta)^{-1}(m'')\mathrm{d}u\mathrm{d}u''\mathrm{d}m\mathrm{d}m'',
$$

where the domain of integration is $(m, m'') \in ((M \cap L)(E) \times L(F))_x \backslash ((M \cap L)(E) \times L(F))$, $u \in N(E) \cap U(E)$, $u'' \in U(F)$, and

$$
f'_K(g) = \int_{K_{H'}} \int_{K_{H''}} f(k_1^{-1}gk_2)\chi_{H'}(k_1)(\chi\eta)^{-1}(k_2)\mathrm{d}k_1\mathrm{d}k_2,
$$

$K_{H'} = \mathrm{GL}_n(\mathfrak{o}_E) \times \mathrm{GL}_n(\mathfrak{o}_E)$ is a maximal open compact subgroup of $H'(F)$ and $K_{H''} = \mathrm{GL}_{2n}(\mathfrak{o}_F)$ is a maximal open compact subgroup of $H''(F)$.

Let us now show that

$$\delta_A : (N \cap U)(E) \times U(F) \to U(E), \quad (u, u'') \mapsto u^{-1} x u'' x^{-1}$$

is bijective and submersive everywhere. Direct computation gives that the tangent map at (u, u'') is given by

$$(\mathfrak{n} \cap \mathfrak{u})(E) \times \mathfrak{u}(E) \to \mathfrak{u}(E), \quad (X, Y) \mapsto -u^{-1} X x u'' x^{-1} + u^{-1} x u'' Y x^{-1}.$$

Since u and u'' are both unipotent, the determinant at any (u, u'') equals the determinant at $(1, 1)$. At the point $(1, 1)$, the tangent can be more explicitly written as

$$M_{n_1 \times n_2}(E) \times M_{n_1 \times n_2}(E) \times M_{2n_1 \times 2n_2}(F) \to M_{2n_1 \times 2n_2}(E),$$

and

$$(X_1, X_2, Y) \mapsto - \begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} + x_1 Y x_2^{-1}.$$

Let $\rho(x)$ be this map and we put

$$\Delta(x) = \delta_{Q(E)}^{\frac{1}{2}}(x) |\det \rho(x)|^{-1}.$$

This will be computed later. We make a change of variable $u^{-1} x u' \mapsto wx$ where $w \in U(E)$. Then we conclude that the orbital integral $O^{G'}(x, f')$ equals

$$(\chi \widetilde{\eta})^{-1}(x) \int \delta_{Q(E)}^{-\frac{1}{2}}(x) \Delta(x) f'_K(m^{-1} w x m'') \chi_{H'}(m) (\chi \eta)^{-1}(m'') \delta_{P(E)}(m)^{-1} \delta_{Q(F)}(m'')^{-1} \mathrm{d}w \mathrm{d}m \mathrm{d}m''.$$

We note that $\delta_{P(E)}(m) = \delta_{Q(E)}(m)^{\frac{1}{2}}$. Therefore a change of variable $w \mapsto mwm^{-1}$ yields that the above integral equals

$$(\chi \widetilde{\eta})^{-1}(x) \int \Delta(x) f'_K(w m^{-1} x m'') (\chi \eta)^{-1}(m') \delta_{Q(E)}(m^{-1} x m'')^{-\frac{1}{2}} \mathrm{d}w \mathrm{d}m \mathrm{d}m''.$$

Put

$$f'^Q(x) = \delta_{Q(E)}(x)^{-\frac{1}{2}} \int_{U''(E)} f'_K(wx) \mathrm{d}w, \quad x \in L(E).$$

Then $f'^Q \in \mathcal{S}(L(E))$. The map $f' \mapsto f'^Q$ is the well-known parabolic descent map. Thus

$$O^{G'}(x, f') = \Delta(x) O^L(x, f'^Q).$$

It remains to compute $\Delta(x)$. The determinant of the map $\rho(x)$ is the same as the determinant of the map

$$M_{2n_1 \times 2n_2}(F) \to M_{n_1 \times n_2}(E) \times M_{n_1 \times n_2}(E), \quad Y \mapsto p(x_1 Y x_2^{-1})$$

where p is the projection of a matrix in $M_{2n_1 \times 2n_2}(E)$ to $M_{n_1 \times n_2}(E) \times M_{n_1 \times n_2}(E)$, the upper right and lower left corner. As we are merely computing determinants, we may pass to the algebraic closure and assume that F is algebraically close. Then E is identified with $F \times F$ and the Galois conjugation exchanges two components in $F \times F$.

Recall that $x\overline{x}^{-1} = s'(\alpha)$ and $\alpha = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}$. One checks readily that $\Delta(x)$ depends only on the conjugacy class of $\alpha_i \overline{\alpha_i}$. Therefore we may assume that α_i is diagonal

$$\alpha_i = \begin{pmatrix} a_1^{(i)} & & \\ & \ddots & \\ & & a_{n_i}^{(i)} \end{pmatrix} \in \mathrm{GL}_{n_i}(F \times F), \quad a_j^{(i)} = (b_j^{(i)}, c_j^{(i)}) \in F^\times \times F^\times.$$

and

$$x_i = (\widetilde{x}_i, 1) \in \mathrm{GL}_{2n_i}(F) \times \mathrm{GL}_{2n_i}(F), \quad \widetilde{x}_i = \begin{pmatrix} B^{(i)} & 1 \\ 1 - B^{(i)}C^{(i)} & -C^{(i)} \end{pmatrix},$$

and

$$B^{(i)} = \begin{pmatrix} b_1^{(i)} & & \\ & \ddots & \\ & & b_{n_i}^{(i)} \end{pmatrix}, \quad C^{(i)} = \begin{pmatrix} c_1^{(i)} & & \\ & \ddots & \\ & & c_{n_i}^{(i)} \end{pmatrix} \in \mathrm{GL}_{n_i}(F).$$

With these choices, the determinant we would like to compute is the product of various determinants of the linear transforms of the form

$$M_{2\times2}(F) \to F \times F \times F \times F, \quad Y \mapsto p_{st}\left(\begin{pmatrix} b_s^{(1)} & 1 \\ 1 - b_s^{(1)}c_s^{(1)} & -c_s^{(1)} \end{pmatrix} Y \begin{pmatrix} b_t^{(2)} & 1 \\ 1 - b_t^{(2)}c_t^{(2)} & -c_t^{(2)} \end{pmatrix}^{-1}, Y \right)$$

where p_{st} is the projection to the upper right and lower left corner, and the product ranges over all $1 \le s \le n_1$ and $1 \le t \le n_2$. Direct computation gives that the determinant is $b_s^{(1)}c_s^{(1)} - b_t^{(2)}c_t^{(2)}$, which in term equals $a_s^{(1)}\overline{a_s^{(1)}} - a_t^{(2)}\overline{a_t^{(2)}}$. According the special form of α_i we took, we have

$$a_i^{(1)}\overline{a_i^{(1)}} - a_j^{(2)}\overline{a_j^{(2)}} = \widetilde{r}_i - \widetilde{t}_j.$$

Moreover $\delta_{Q(E)}(x) = 1$. It follows that $\Delta(x) = \lambda'$.

We summarize the above computation in the following proposition.

Proposition 7.2. *Let the notation be as above. Then*

$$O^{G'}(x, f') = \lambda' \cdot O^L(x, f'^Q).$$

7.4. **Reduction to the elliptic case.** Let us come back to the setup of Theorem 7.1. Assume that $\alpha\overline{\alpha} \in \mathrm{GL}_n(E)$ is regular semisimple but not elliptic (in the usual sense). Then we can find positive integers n_1, n_2 with $n_1 + n_2 = n$, $P = MN$ be the standard blocked upper triangular parabolic subgroup of GL_n corresponding to this partition, and α is twisted conjugate to $\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \in M(E)$.

Since $s'(\alpha)$ and $g(\beta)$ match, we may find β is twisted conjugate to $\begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \in M(E)$, $s'(\alpha_i) \in S'_{2n_i}$ matches $g(\beta_i) \in \mathrm{GL}_{2n_i}(E)$, $i = 1, 2$.

Put $f' = \mathbf{1}_{\mathrm{GL}_{2n}(\mathfrak{o}_E)}$ in Proposition 7.2. Since $f'^Q = \mathbf{1}_{\mathrm{GL}_{2n_1}(\mathfrak{o}_E)} \otimes \mathbf{1}_{\mathrm{GL}_{2n_2}(\mathfrak{o}_E)}$, and

$$O^{G'}(x, \mathbf{1}_{\mathrm{GL}_{2n}(\mathfrak{o}_E)}) = O^{S'}(x\overline{x}^{-1}, \mathbf{1}_{S'(\mathfrak{o}_F)}),$$

we conclude

$$O^{S'}(s'(\alpha), \mathbf{1}_{S'(\mathfrak{o}_F)}) = \lambda' O^{S'_{2n_1}}(s'(\alpha_1), \mathbf{1}_{S'_{2n_1}(\mathfrak{o}_F)}) O^{S'_{2n_2}}(s'(\alpha_2), \mathbf{1}_{S'_{2n_2}(\mathfrak{o}_F)}).$$

Let $r_1, \cdots, r_{n_1}$ be the eigenvalues of $\beta_1 \overline{\beta_1}$ and $s_1, \cdots, s_{n_2}$ be the eigenvalues of $\beta_2 \overline{\beta_2}$ (in some fixed algebraic closure of F). Put

$$\lambda = \frac{|\det(1 - \beta_1 \overline{\beta_1})|^{n_2} |\det(1 - \beta_2 \overline{\beta_2})|^{n_1}}{|\prod_{1 \leq i \leq n_1, 1 \leq j \leq n_2}(r_i - s_j)|}.$$

Then one checks that $\lambda \in F$. By [Guo96, Proposition 2.2] we have

$$O^{\mathrm{GL}_{2n}}(g(\beta), \mathbf{1}_{\mathrm{GL}_{2n}(\mathfrak{o}_F)}) = \lambda O^{\mathrm{GL}_{2n_1}}(g(\beta_1), \mathbf{1}_{\mathrm{GL}_{2n_1}(\mathfrak{o}_F)}) O^{\mathrm{GL}_{2n_2}}(g(\beta_2), \mathbf{1}_{\mathrm{GL}_{2n_2}(\mathfrak{o}_F)}).$$

Since $s'(\alpha_i)$ and $g(\beta_i)$ match, $i = 1, 2$, the elements $\beta_i \overline{\beta_i}$ and $-(1 - \alpha_i \overline{\alpha_i})(\alpha_i \overline{\alpha_i})^{-1}$ have the same characteristic polynomial. It follows that

$$\lambda' = \lambda.$$

Then Theorem 7.1 follows by induction on n. This finishes the proof of Theorem 7.1.

APPENDIX A. CONVERGENCE OF THE ELLIPTIC PART

The goal of this appendix is to explain the absolute convergence of the elliptic part of the relative trace formula. We will prove

(A.1)
$$\int_{H'(F) \backslash H'(\mathbb{A}_F)^1} \sum_{x \in S'(F)_{\mathrm{ell}}} |f'(h^{-1} x \overline{h})| \mathrm{d}h$$

is convergent for all $f \in \mathcal{S}(S'(\mathbb{A}_F))$, where

$$H'(\mathbb{A}_F)^1 = \{(h_1, h_2) \in H'(\mathbb{A}_F) \mid |\det h_1 h_2| = 1\}.$$

This implies the absolute convergence of (2.2). The proof of the absolute convergence of (2.1) is similar.

Let P_0 be the usual upper triangular Borel subgroup of GL_n and $P' = \mathrm{Res}_{E/F} P_0 \times P_0$ be a minimal parabolic subgroup of H'. Let c be a real number with $0 < c < 1$ and T_c the subset of the diagonal torus in $\mathrm{GL}_{2n}(\mathbb{R})$ consisting of

$$\{(a_1, \cdots, a_n, b_1, \cdots, b_n) \in (\mathbb{R}_{>0})^{2n} \mid a_i a_{i+1}^{-1} \geq c, \ b_i b_{i+1}^{-1} \geq c, \ a_1 \cdots a_n b_1 \cdots b_n = 1\}.$$

Let T_c be diagonally embedded in $H'(F_\infty)$ and identify it with its image. Fix a maximal compact subgroup $\mathcal{K}$ of $H'(\mathbb{A}_F)$. Then reduction theory gives that there is a compact subgroup $\omega \subset P'(\mathbb{A}_F)$, such that $H'(\mathbb{A}_F)^1 = H'(F)\mathcal{G}$ and

$$\mathcal{G} = \{pak \mid p \in \omega, \ a \in T_c, \ k \in \mathcal{K}\}.$$

Thus we only need to prove that

$$\int_\omega \int_{T_c} \int_\mathcal{K} \sum_{x \in S'(F)_{\mathrm{ell}}} |f'((pak)^{-1} x (\overline{pak}))| \delta_{P'}(a)^{-1} \mathrm{d}k \mathrm{d}a \mathrm{d}p$$

is absolutely convergent. By the definition of T_c, there is a compact subset Ω of $H'(\mathbb{A}_F)$ such that if $p \in \omega$, $a \in T_c$, $k \in \mathcal{K}$, then $a^{-1}pak \in \Omega$. It follows that we only need to prove that

$$\int_{T_c} \sum_{x \in S'(F)_{\mathrm{ell}}} |f'(a^{-1}xa)| \delta_{P'}(a)^{-1} \mathrm{d}a$$

is absolutely convergent for all Schwartz functions f' on $S'(\mathbb{A}_F)$. It is enough to consider $f' = \otimes_v f'_v$, where f'_v is a Schwartz function on $S'(F_v)$. Since f'_v is compactly supported if $v \nmid \infty$ and $T_c \subset H'(F_v)$, we just need to prove that

(A.2) $$\int_{T_c} \sum_{x \in S'(L)_{\mathrm{ell}}} |f'_\infty(a^{-1}xa)| \delta_{P'}(a)^{-1} \mathrm{d}a$$

is absolutely convergent for any Schwartz function f'_∞ on $S'(F_\infty)$ and any fractional ideal L of $\mathfrak{o}_F$. Note that L is discrete in F_∞.

We fix some notation. Let $v \mid \infty$ be an infinite place we write $|\cdot|$ for the usual absolute value. If $x = (x_v) \in F_\infty$ we write $|x|$ for $\max_{v \mid \infty} |x_v|$. If $X = (x_{ij}) \in M_n(F_\infty)$, then we write $\|X\| = \max_{ij} |x_{ij}|$.

Let us divide the integral into two pieces depending on $a_1 \cdots a_n > 1$ or not. We will treat the case $a_1 \cdots a_n > 1$. The case $a_1 \cdots a_n < 1$ can be handled in exactly the same way by noting that $b_1 \cdots b_n > 1$ under this assumption.

From now on assume that $a_1 \cdots a_n > 1$. Then $b_1 \cdots b_n = (a_1 \cdots a_n)^{-1} < 1$.

Since L is a fractional ideal, there is a constant $c_L > 0$ such that if $x \in S'(L)$ and u is a nonzero entry of x then $|u| \geq c_L$. This is where the discreteness of L in F_∞ is used.

We write $x \in S'(F_\infty)$ as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Fix a positive polynomial P_1 such that

$$P_1(x) \geq \max\{\|A\|, \|B\|, \|C\|, \|D\|\}.$$

Here polynomial means that we view $S'(F_\infty)$ as a real manifold and a P_1 is a real positive polynomial, in other worlds, if a_{ij} is an entry of A, then both a_{ij} and $\overline{a_{ij}}$ might appear in the polynomial P_1.

If $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S'(F_\infty)_{\mathrm{ell}}$, we write $A = (x_{ij})$. Since the characteristic polynomial of $A\overline{A}$ is irreducible over F, for every $i_0 = 1, \cdots, n-1$, there is a $j \geq i_0 + 1$ and $i \leq i_0$ such that $x_{ji} \neq 0$ (for otherwise A is contained in a proper parabolic subgroup of $\mathrm{GL}_n(E)$). Thus $|x_{ji}| \geq c_L$. Something similar holds for the entries of D. This is where the condition "elliptic" is used.

By the choice of P_1 we have

$$P_1(a^{-1}xa) \geq |a_j^{-1} x_{ji} a_i| \geq c_L a_i a_j^{-1}.$$

Since $a \in T_c$ we have

$$a_i \geq c a_{i+1} \geq \cdots \geq c^{i_0 - i} a_{i_0}, \quad a_j^{-1} \geq c a_{j-1}^{-1} \geq \cdots \geq c^{j-i_0-1} a_{i_0+1}^{-1}.$$

Therefore

$$(A.3) \qquad P_1(a^{-1}xa) \geq c_L c^{j-i-1} a_{i_0} a_{i_0+1}^{-1} \geq c_L c^{n-2} a_{i_0} a_{i_0+1}^{-1}.$$

Note that we used the fact that $0 < c < 1$. So we obtain

$$a_n^{-1} = (a_1 \cdots a_n)^{-\frac{1}{n}} \prod_{i=1}^{n-1} (a_i a_{i+1}^{-1})^{\frac{i}{n}} \leq (a_1 \cdots a_n)^{-\frac{1}{n}} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{\frac{n-2}{2}},$$

and

$$a_1 = (a_1 \cdots a_n)^{\frac{1}{n}} \prod_{i=1}^{n-1} (a_i a_{i+1}^{-1})^{\frac{n-i}{n}} \leq (a_1 \cdots a_n)^{\frac{1}{n}} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{\frac{n-2}{2}},$$

Similarly by considering D, we conclude

$$(A.4) \qquad P_1(a^{-1}xa) \geq c_L c^{n-2} b_{i_0} b_{i_0+1}^{-1},$$

and

$$b_n^{-1} \leq (b_1 \cdots b_n)^{-\frac{1}{n}} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{\frac{n-2}{2}}, \quad b_1 \leq (b_1 \cdots b_n)^{\frac{1}{n}} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{\frac{n-2}{2}}.$$

For any $i, j = 1, \cdots, n$ we also have

$$P_1(a^{-1}xa) \geq |a_i^{-1} x_{ij} a_j|,$$

and thus

$$(A.5)$$
$$|x_{ij}| \leq a_i a_j^{-1} P_1(a^{-1}xa) \leq c^{-(i-1)-(n-j)} a_1 a_n^{-1} P_1(a^{-1}xa) \leq c^{-(2n-2)} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{n-2}.$$

Write $C = (z_{ij})$, $D = (w_{ij})$. Similar considerations also give

$$(A.6)$$
$$|z_{ij}| \leq (a_1 \cdots a_n)^{-\frac{1}{n}} (b_1 \cdots b_n)^{\frac{1}{n}} c^{-(2n-2)} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{n-2}$$
$$= (a_1 \cdots a_n)^{-\frac{2}{n}} c^{-(2n-2)} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{n-2},$$

and

$$(A.7) \qquad |w_{ij}| \leq a_i a_j^{-1} P_1(a^{-1}xa) \leq c^{-(2n-2)} \left(c_L^{-1} c^{-(n-2)} P_1(a^{-1}xa) \right)^{n-2}.$$

To summarize, multiplying the inequalities (A.5), (A.6) and (A.7), we obtain a positive polynomial function P on $S'(F_\infty)$ such that

$$(A.8) \qquad \|A\|\|C\|\|D\| \leq (a_1 \cdots a_n)^{-\frac{2}{n}} P(a^{-1}xa).$$

Let us now fix a positive homogeneous polynomial Q in $M_n(F_\infty)$ of a large degree M. Consider

$$\phi(x) = f'_\infty(x) Q(B).$$

This is still a Schwartz function on $S'(F_\infty)$. Then

$$(A.2) = \int \sum_{x \in S'(L)_{\text{ell}}} \phi(a^{-1}xa)(a_1 \cdots a_n)^{2M} Q(B)^{-1} \delta_{P'}(a)^{-1} \mathrm{d}a.$$

55

where as before we write each $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Since ϕ is Schwartz, it is bounded by the reciprocal of any polynomial and in particular by P^{-N} when N is large enough, thus by (A.8) we have

$$(\text{A.2}) \leq \int \sum_{x \in S'(L)_{\text{ell}}} (\|A\|\|C\|\|D\|)^{-N} (a_1 \cdots a_n)^{\frac{-2N}{n}} (a_1 \cdots a_n)^{2M} Q(B)^{-1} \delta_{P'}(a)^{-1} \mathrm{d}a$$

$$= \sum_{x \in S'(L)_{\text{ell}}} (\|A\|\|C\|\|D\|)^{-N} Q(B)^{-1} \times \int (a_1 \cdots a_n)^{\frac{-2N}{n}} (a_1 \cdots a_n)^{2M} \delta_{P'}(a)^{-1} \mathrm{d}a.$$

Here N is a sufficiently large real number, and the integration is over $a \in T_c$ and $a_1 \cdots a_n > 1$. The point is that the variables in the integral, i.e. $a_1, \cdots, a_n, b_1, \cdots, b_n$, and the variables in the sum, i.e. $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, are separated. Thus when $N >> M >> 0$, both the sum and the integral are convergent. This proves the convergence of (A.2) and hence the absolute convergence of (2.2).

APPENDIX B. ELLIPTIC REPRESENTATIONS

The goal of this appendix is to sketch a proof of Proposition 3.4. To simplify notation, we fix in this subsection a nonarchimedean nonsplit place v of F and suppress it from all notation. Thus F stands for a nonarchimedean local field of characteristic zero. To shorten notation we also write H for its group of F-points $H(F)$. The equalities in the proof usually depend on the choice of the measures. But such choices are not essential to the final result. Thus we should interpret the equalities in the proof as equalities up to a nonzero constant depending only the choice of the measures.

B.1. Results on orbital integrals. First we need some results on the nilpotent orbital integrals and Shalika germ. let $\mathfrak{s}$ be the tangent space of S at 1, with an action of H by conjugation. An element $x \in \mathfrak{s}$ is called regular semisimple if H_x is a torus of dimension n, and it is called elliptic if in addition H_x is an elliptic torus modulo the split center of H. Regular semisimple orbital integrals has been defined and studied in [Zha15a]. An H-orbit in $\mathfrak{s}$ is called nilpotent if its closure contains 0. Nilpotent orbital integrals have been defined in [Guo98]. In particular if $\mathcal{O}$ is an nilpotent orbit in $\mathfrak{s}$, it is proved in [Guo98] that the integral

$$\int_{\mathcal{O}} f(x) \mathrm{d}x, \quad f \in \mathcal{S}(\mathfrak{s}),$$

is absolutely convergent, where $\mathrm{d}x$ is an invariant Radon measure on $\mathcal{O}$. Moreover it is proved that the Fourier transform $\widehat{\mu_{\mathcal{O}}}$ of the distribution $\mu_{\mathcal{O}}$ is a locally integrable function on $\mathfrak{s}$. If $\mathcal{O} = \{0\}$ is the smallest nilpotent orbit, then obviously $\widehat{\mu_{\mathcal{O}}}(X)$ is a nonzero constant. More importantly $\mu_{\mathcal{O}}$ and $\widehat{\mu_{\mathcal{O}}}$ have the following homogeneity property. If $t \in F^\times$, then

$$\mu_{\mathcal{O}}(f_t) = |t|^{\dim \mathcal{O}} \mu_{\mathcal{O}}(f), \quad f_t(X) = f(t^{-1}X).$$

This follows from the explicit formula for $\mu_{\mathcal{O}}$ given in [Guo98, Proposition 5.1]. Taking Fourier transform we conclude that

$$\widehat{\mu_{\mathcal{O}}}(f_t) = |t|^{2n^2 - \dim \mathcal{O}} \widehat{\mu_{\mathcal{O}}}(f).$$

The most important point is that $\dim \mathcal{O} < n^2$ for all $\mathcal{O}$, and thus

(B.1) $$\dim \mathcal{O} < 2n^2 - \dim \mathcal{O}'$$

for any two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$.

As in the classical situation of Harish-Chandra, we have the Shalika germs. Let $\exp : \mathfrak{s} \to S$ be the exponential map, defined in an H-invariant neighbourhood $0 \in \mathfrak{s}$. For any $f \in \mathcal{S}(G)$, we define in an H-invariant neighbourhood of $0 \in \mathfrak{s}$ a function $f_\natural$ by requiring that

$$\int_H f(gh)\chi(gh)^{-1}\mathrm{d}h = f_\natural(X)$$

if $g\theta(g)^{-1} = \exp X$. There is a unique H-invariant real valued function $\Gamma_{\mathcal{O}}$ on the regular semisimple locus of $\mathfrak{s}$ for each nilpotent orbit $\mathcal{O}$ with the following properties.

(1) For any $f \in \mathcal{S}(\mathfrak{s})$, there is an H-invariant neighbourhood U_f of $0 \in \mathfrak{s}$ such that

(B.2) $$O^G(g, f) = \sum_{\mathcal{O}} \Gamma_{\mathcal{O}}(X)\mu_{\mathcal{O}}(f_\natural).$$

for all regular semisimple $g \in U_f$, such that $g\theta(g)^{-1} = \exp(X)$.

(2) For all $t \in F^\times$ and all regular semisimple X, we have

$$\Gamma_{\mathcal{O}}(tX) = |t|^{-\dim \mathcal{O}}\Gamma_{\mathcal{O}}(X).$$

Lemma B.1. *The Shalika germs $\Gamma_{\mathcal{O}}$ are linearly independent. They are not identically zero in any neighbourhood of 0. If $\mathcal{O} = \{0\}$ the minimal nilpotent orbit, then $\Gamma_0(X) = 0$ if X is not elliptic in $\mathfrak{s}$.*

Proof. The linear independence is proved by exactly the same argument as in the classical case of Harish-Chandra. The key to this argument is the inequality (B.1), and the rest of the argument is essentially formal, cf. [Kot05, Section 27] and [Xue22, Section 7]. The fact that $\Gamma_0(X) = 0$ if X is not elliptic is proved using parabolic descent [Zha15a, Subsection 6.1] and the homogeneity property of $\Gamma_{\mathcal{O}}$'s. $\square$

B.2. **Characters of supercuspidal representations.** Now we recall that by [BP18, Proposition 4.2.1], in the case π being supercuspidal, up to some nonzero constant depending only on the choice of the measures and the linear form ℓ, we have

(B.3) $$\ell(v)\overline{\ell(w)} = \int_{Z \backslash H} \langle v, \pi(h)w \rangle \chi(h)^{-1}\mathrm{d}h$$

for all $v, w \in \pi$. Thus if $\varphi \in \mathcal{S}(G)$ then

$$J_\pi(\varphi) = \sum_v \int_{Z \backslash H} \langle \pi(\varphi)v, \pi(h)v \rangle \chi(h)^{-1}\mathrm{d}h,$$

where the sum runs over an orthonormal basis of π. By [RR96, Corollary 5.2] the distribution J_π agrees with a locally constant function on the regular semisimple locus in G. We denote this function by Θ_π.

Recall from (2.5) that we have defined a transfer factor

$$\kappa^G(g) = \chi(A), \quad g^{-1} = \begin{pmatrix} A & * \\ * & * \end{pmatrix}$$

for all regular semisimple $g \in G$. We put $\widetilde{\Theta}_\pi(g) = \kappa^G(g)\Theta_\pi(g)$. Then $\widetilde{\Theta}_\pi$ is left and right H-invariant, and we can view it as a function on S which is H-conjugate invariant. By [RR96, Theorem 7.11], if X is in a small neighbourhood of $0 \in \mathfrak{s}$, $g \in G$, $g\theta(g)^{-1} = \exp X$, we have

(B.4) $$\widetilde{\Theta}_\pi(g) = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu_{\mathcal{O}}}(X).$$

The case treated in [RR96] does not involve the character, but the same argument goes through without change in our setup.

Lemma B.2. *Let $v, w \in \pi$, and $f(g) = \langle v, \pi(g)w \rangle$ be the matrix coefficient. Then we have*

(B.5) $$\kappa^G(g)O^G(g, f) = \widetilde{\Theta}_\pi(g) \int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h,$$

for all elliptic g in G.

Proof. It is enough to prove that for any $\varphi \in \mathcal{S}(G)$ supported in the elliptic locus, we have

(B.6) $$\int_G \varphi(g)\kappa^G(g)O^G(g, f)\mathrm{d}g = J_\pi(\varphi\kappa^G) \int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h.$$

Though κ^G is not defined on all G, as φ is locally constant and compactly supported in the elliptic locus, $\varphi\kappa^G \in \mathcal{S}(G)$ and $J_\pi(\varphi\kappa^G)$ makes sense.

Let us first note that because $(H \times H)_g$ is an elliptic torus modulo the center of G, up to some nonzero constant depending only on the choice of the measures, the orbital integral of f equals

$$\kappa^G(g) \int_{Z \backslash H \times Z \backslash H} f(h_1^{-1}gh_2)\chi(h_1^{-1}h_2)^{-1}\mathrm{d}h_1\mathrm{d}h_2.$$

As φ is supported on the elliptic locus, we have

$$\int_G \varphi(g)\kappa^G(g)O^G(g, f)\mathrm{d}g = \int_G \int_{Z \backslash H \times Z \backslash H} \varphi(g)\kappa^G(g)f(h_1^{-1}gh_2)\chi(h_1^{-1}h_2)^{-1}\mathrm{d}h_1\mathrm{d}h_2\mathrm{d}g.$$

The right hand of this integral is absolutely convergent. Thus we may change the order of integration and conclude that this integral equals

(B.7) $$\int_{Z \backslash H \times Z \backslash H} \langle v, \pi(h_1^{-1})\pi(\overline{\varphi\kappa^G})\pi(h_2)w \rangle \chi(h_1^{-1}h_2)^{-1}\mathrm{d}h_1\mathrm{d}h_2.$$

Since π is admissible, we may find elements $v_1, \cdots, v_r$ and $w_1, \cdots, w_r$ in π such that

$$\pi(\varphi\kappa^G)v_0 = \sum_{i=1}^r \langle v_0, v_i \rangle w_i,$$

for all $v_0 \in \pi$. It then follows by (B.3) that

$$(\text{B.7}) = \sum_{i=1}^{r} \ell(v)\overline{\ell(v_i)}\ell(w_i)\overline{\ell(w)}.$$

We also have

$$J_\pi(\varphi\kappa^G) = \sum_u \ell(\pi(\varphi\kappa^G)u)\overline{\ell(u)} = \sum_u \sum_{i=1}^{r} \langle u, v_i \rangle \ell(w_i)\overline{\ell(u)} = \sum_{i=1}^{r} \ell(w_i)\overline{\ell(v_i)}.$$

Thus (B.6) follows by another application of (B.3). $\qquad\square$

Proof of Proposition 3.4. Let X be in a small neighbourhood of $0 \in \mathfrak{s}$, $g \in G$, $g\theta(g)^{-1} = \exp X$. The character expansion (B.4) gives

$$\widetilde{\Theta}_\pi(g) = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu_{\mathcal{O}}}(X).$$

Note that X is elliptic if and only if g is elliptic. Since $\mathcal{O} = \{0\}$ is the only nilpotent orbit with $\widehat{\mu_{\mathcal{O}}}(tX) = \widehat{\mu_{\mathcal{O}}}(X)$ for all $X \in \mathfrak{s}$ and $t \in F^\times$, to show that $\widetilde{\Theta}_\pi(g) \neq 0$ for some elliptic $g \in G$ which is sufficiently close to 1, we only need to show that $c_0 \neq 0$.

We find a matrix coefficient f such that

$$\int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h \neq 0.$$

For this f we consider the expansion of both sides of (B.5) when g is close to 1 and is elliptic. We have

$$\sum_{\mathcal{O}} \Gamma_{\mathcal{O}}(X)\mu_{\mathcal{O}}(f_\natural) = \sum_{\mathcal{O}} c_{\mathcal{O}}\widehat{\mu_{\mathcal{O}}}(X) \times \int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h.$$

The only terms on both sides of the expansion that are invariant under the scaling $X \to tX$ are those corresponding to $\mathcal{O} = 0$. Thus by the homogeneity property of $\Gamma_{\mathcal{O}}$ and $\widehat{\mu_{\mathcal{O}}}$, we conclude that

$$\Gamma_0(X)\mu_0(f_\natural) = c_0\widehat{\mu_0}(X) \times \int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h.$$

By our choice of f we have

$$\mu_0(f_\natural) = \int_{Z \backslash H} f(h)\chi(h)^{-1}\mathrm{d}h \neq 0.$$

If $c_0 = 0$, then $\Gamma_0(X) = 0$ if X is elliptic in a neighbourhood of 0. By Lemma B.1, $\Gamma_0(Y) = 0$ if Y is not elliptic and hence is identically zero in a neighbourhood of 0. This is impossible by Lemma B.1. Therefore $c_0 \neq 0$. $\qquad\square$

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