INTRODUCTION TO THE SPECTRAL THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

Course Description: This course aims to introduce graduate students to methods in the geometry of eigenvalues, focusing on isoperimetric inequalities and universal bounds. Isoperimetric inequalities fall in the following categories:

- Spectral inequalities related to bounds for eigenvalues of the Laplacian, or more generally the Laplace-Beltrami operator, under some geometric constraint; bounds for other physical quantities such as torsional rigidity (the St. Venant problem), or sharp inequalities that relate various physical parameters in the tradition of the classical work of Pólya and Szegő (e.g. Rayleigh-Faber-Krahn, Payne-Pólya-Weinberger, Szegő-Weinberger, and Cheeger's inequalities, etc.)
- Sharp inequalities for integral functionals ("geometric inequalities"); sharp constants for the Sobolev inequality, the Hardy-Littlewood-Sobolev inequality, and Lieb-Thirring inequalities fall into this category;
- Classical isoperimetric inequalities and their modern counterparts; the study of extremizing geometries under various geometric constraints.

Rearrangement techniques (symmetric decreasing rearrangement, weighted-decreasing rearrangement, Steiner, Schwarz, etc.) as well as fundamental results by Pólya-Szegő, Talenti, and followers will be explored. We will also provide a comprehensive survey of universal bounds for eigenvalues which provide counterparts to the classical works of Weyl, Berezin and Li-Yau, and others.

The course will also survey recent developments focusing on quantitative isoperimetric inequalities–specifically involving asymmetry. These have applications in relation to the shape of liquid drops and crystals in the small mass regime. Related techniques from variational calculus and pertubation theory based on recent works will also be discussed. These will focus on the computation of eigenvalues and the torsional rigidity of a domain. This latter physical quantity is defined by a Rayleigh-Ritz principle similar in form to that defining the eigenvalues but is simpler to deal with.

Textbook(s): "Analysis" by Lieb & Loss; Lecture Notes developed by A. Burchard, R. Laugesen; original fundamental papers by Talenti, Ashbaugh-Benguria, Harrell-Stubbe, I. Chavel, C. Bandle, Payne, Weinberger, etc., as well as recent papers by Figalli, Maggi, Pratelli, Cianchi, van den Berg, Bhattacharya-Weitsmann, and others.

Prerequisites: Core Math. or Applied Math. courses; some knowledge of PDEs is recommended, but is not required.