INTRODUCTION TO INFORMATION THEORY—a topics course proposal

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Main text: "Elements of Information Theory" by T. M. Cover and J. A. Thomas, 2nd ed. (Wiley 2006).

Intended audience: The course will be an introduction to mathematical concepts and tools of information theory. It is intended for a broad audience of graduate students in Mathematics, Applied Mathematics, Physics, Computer Science, Optical Science, Computer and Systems Engineering and whoever else may be interested. The exposition will be mathematical and rigorous where possible, but the focus will be on concepts and applied context, not on technically difficult mathematics. Accordingly, I am assuming no background beyond Applied Mathematics or Mathematics core courses. Students from other programs are encouraged to contact me to make sure they have enough preparation, but, as always, interest and enthusiasm are more important than formal course credits.

Wide reach of the subject: Information theory was born with Claude Shannon's seminal work in 1948. While his original motivation was communication (sending information) over a noisy channel, the theory he created turned out to be so rich and versatile that today it appears in contexts and applications ranging from Riemannian geometry and theory of stochastic processes, through data science, algorithmic complexity and statistical physics, to thermodynamics of dissipative structures, molecular biology and origins of life. Selected readings illustrating these connections will be recommended during the course, which will aim to show the breadth of current application of Shannon's original ideas and to present their conceptual flexibility. See also the list of books on the next page.

An approximate list of topics I am planning to discuss:

1. Basic quantities: information, entropy, mutual information, relative entropy—this establishes the mathematical language of the theory.

2. Asymptotic equipartition property—this is Shannon's brilliant insight and the cornerstone of the theory.

3. Entropy in stochastic processes—we will mostly discuss the Markov chain case, but this is a point of departure to the exciting recent developments in statistical physics—see point 8 below.

4. Mathematics of data compression—these applications probably needs no advertisement.

5. Communication channels—this theme is closest to Shannon's original motivation, and at least as important now as at his time

6. Applications to statistics: Fisher information, Cramér-Rao inequality, hypothesis testing.

7. Algorithms, computation, complexity, foundations of mathematics.

8. Statistical physics.

There is life beyond the Fall of 2022

Points 7 and 8 above are whole worlds and will be treated briefly. I will just mention the relations to Turing computability, algorithmic complexity, Gödel's incompleteness theorem, Chaitin's Ω number (part 7) and irreversibility, Second Law of Thermodynamics, Maxwell's demon, fluctuation theorems, entropy production (part 8). I hope to convince you that these fascinating topics are worth pursuing further—for example, in the form of a topics course in the Spring of 2023 on Kolmogorov complexity, featuring statistical mechanics.

Selected readings to wet your appetite: I am not going to teach extensive parts of the books on this list. It is here to show you the scope of themes related to information theory. This course will give you a conceptual and technical foundation to study these more advanced topics.

P. Billingsley: Ergodic Theory and Information. Wiley 1965

L. Brillouin: Science and Information Theory. Dover 2013

H. Haken: Synergetics. Introduction and Advanced Topics. Springer 2004

D. Kondepudi, I. Prigogine: Modern Thermodynamics. From Heat Engines to Dissipative Structures. Wiley 1998

W. R. Loewenstein: The Touchstone of Life. Molecular Information, Cell Communication and the Foundations of Life. Oxford University Press 1999

M. Li, P. Vitányi: Introduction to Kolmogorov Complexity and Its Applications. 3rd ed., Springer 2008

D.J.C. MacKay: Information Theory, Inference, and Learning Algorithms. Cambridge University Press 2003

A. Shen, V.A. Uspensky, N. Vereshchagin: Kolmogorov Complexity and Algorithmic Randomness. American Mathematical Society 2017

H. P. Yockey: Information Theory and Molecular Biology. Cambridge University Press 1992