ALGEBRA QUALIFYING EXAM
FALL 2009

- Do any one of the problems nA or nB where $n = 1, 2, 3, 4, 5$.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.

1A: Let A be an $n \times n$ matrix with complex entries. Assume that A is nilpotent (i.e. $A^m = 0$ for some $m \geq 1$). Show that that the trace of A is zero.

1B: Let A be the following n by n integer matrix for $n \geq 3$:

$$A := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

Compute the determinant of A.

2A: Let G be a finite simple group of order 168. Show that there exists an injective homomorphism $G \hookrightarrow S_8$. Is there an injective homomorphism $G \hookrightarrow S_6$?

2B: Let G be a finite group, p a prime, N a normal subgroup of G and P be a Sylow p-subgroup of N. Show that $G = N_G(P) \cdot N$.

3A: Let $R = \mathbb{C}[x]$. Determine all simple modules over R up to isomorphism.

3B: Let R be the subring of $\mathbb{Q}$ consisting of all $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and b is odd. Show that R is a principal ideal domain and determine all ideals of R.

4A: Show that for every $n \geq 1$, there exists an irreducible polynomial $f_n(X) \in \mathbb{Q}[x]$, of degree n. Show that this implies that $\bar{\mathbb{Q}}/\mathbb{Q}$ is not finite.

4B: Let $\alpha := \sqrt{2 + \sqrt{2}} \in \mathbb{C}$. Determine the splitting field $K \subseteq \mathbb{C}$ of the minimal polynomial of α (over $\mathbb{Q}$); determine the Galois group of $K/\mathbb{Q}$ and all subfields of K.

5A: Find all semi-simple rings of order 1200.

5B: Let G be an abelian group with generators x, y, z, t and with defining relations $xy = z$, $yz = t$, $zt = x$, and $tx = y$. Write G as a direct product of cyclic groups and determine whether there is a group homomorphism of G onto $\mathbb{Z}$.