ALGEBRA QUALIFYING EXAM
FALL 2011

- Do any one of the problems nA or nB where $n = 1, 2, 3, 4, 5$.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.
- Each problem is worth 25 points.
- Show all steps and details; say what you mean, mean what you say.

1A: Find all the similarity classes of matrices A in $\mathrm{GL}_2(\mathbb{Q})$ that satisfy $A^5 = A$.

1B: Let K be an algebraically closed field, $n \in \mathbb{N}$, and let A, B be n by n matrices with entries in K such that $AB = BA$. Show that there is an invertible n by n matrix T with entries in K such that $A' := TAT^{-1}$ and $B' := TBT^{-1}$ are both upper triangular (i.e., the entries $A'_{i,j}$ $B'_{i,j}$ are zero for $1 \leq j < i \leq n$).

2A: Suppose $p \geq 3$ and $2p - 1$ are both prime numbers (for example, $p = 3, 7, 19, 31, \ldots$). Prove, or disprove by example, that every group of order $p(2p - 1)$ is abelian.

2B: Let F be a free group on two generators x, y and let N be the minimal normal subgroup of F containing $x^2, y^5, xyxy$. Show that F/N is isomorphic to the dihedral group of order 10.

3A: Let R be a ring, M an R-module and N an R-submodule of M. Show that M satisfies the ascending chain condition if and only if M/N and N satisfy the ascending chain condition.

3B: Let R be the subring of the polynomial ring $\mathbb{C}[t]$ consisting of polynomials of the form $a_0 + a_2t^2 + a_4t^4 + a_5t^5 + a_6t^6 + \ldots + a_nt^n$ (i.e., the coefficients of t and t^3 are 0). Prove or disprove: R is a unique factorization domain.

4A: Let E be the splitting field over $\mathbb{Q}$ of $(X^2 - 3)(X^2 - 5)(X^2 - 7)$. Find all the subfields K such that $\mathbb{Q} \subseteq K \subseteq E$. Find an element $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$.

4B: Consider the polynomial $g = X^4 + X^3 + X^2 + X + 1$ over the field with two elements, $\mathbb{F}_2$.

(a) Show that g is irreducible over $\mathbb{F}_2$.

(b) Let K be a splitting field for g over $\mathbb{F}_2$ and let $r \in K$ be a root of g. Write g as a product of irreducible polynomials over $\mathbb{F}_2(r)$.

(c) Show that $K = \mathbb{F}_2(r)$.

(d) Determine the Galois group of K over $\mathbb{F}_2$.

5A: Prove that there is an isomorphism of rings $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y] \cong \mathbb{C}[X, Y]$.

5B: Let V denote the $\mathbb{Z}$-module $\mathbb{Z}^2$ and let $u_1, u_2, u_3 \in V$ be $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $u_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Write $V \otimes V / \langle u_1 \otimes u_2, u_2 \otimes u_3, u_1 \otimes u_3 \rangle_{\mathbb{Z}}$ as a direct sum of cyclic modules.