ALGEBRA QUALIFYING EXAMINATION

AUGUST 2013

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say.

1A. Let $A \in \mathrm{GL}_n(\mathbb{Z})$ be an $n \times n$ invertible matrix with integer entries. Suppose that A has order p^k for some prime p and integer $k \geq 1$. Prove that $n \geq p^{k-1}(p-1)$.

1B. Find any orthogonal matrix $A \in \mathrm{GL}_5(\mathbb{R})$ such that $A + A^{-1} = (b_{ij})$ is diagonal with entries $b_{11} = -2$, $b_{22} = b_{33} = 1$, and $b_{44} = b_{55} = 2$.

2A. Let D_n be the dihedral group of order $2n$ for some even $n \geq 2$.
  i) Describe the group $\mathrm{Inn}(D_n)$ of inner automorphisms of D_n as a quotient of D_n.
  ii) Describe a normal subgroup of $\mathrm{Aut}(D_n)$ that is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and determine which of its elements are inner automorphisms.

2B. Prove that A_6 has no subgroup of index 5.

3A. Let R and S be nonzero rings with $M_n(R) \simeq M_m(S)$ (as rings) for integers n, m.
  i) If R and S are commutative, prove that $m = n$ and $R \simeq S$.
  ii) Give an example to show that the conclusions of i) fail if we do not assume that R and S are commutative.

3B. Show that every subring of $\mathbb{Q}$ is a PID, and moreover show that there is a canonical bijection between the set of these subrings and the set of sets of prime numbers.

4A. Let F be a field, and let K, L, and M denote finite extensions of F with $K, L \subseteq M$ and $K \cap L = F$.
  i) Show by way of example that $[KL : F]$ need not equal $[K : F][L : F]$.
  ii) Suppose that K/F is Galois. Show that $[KL : F] = [K : F][L : F]$.

4B. Let $f(x) = x^4 - 2$. For each of the following fields F, find an an isomorphism between the Galois group of f over F and a (standard) abstract group.
  i) $F = \mathbb{Q}$.
  ii) $F = \mathbb{F}_7$.

5A. Let k be a field and $R = k[x]$. Let $M = R^3$, and let N be the free R-submodule of M spanned by (x^2, x^3, x^4) and $(1, x+1, x^2)$. Express the quotient M/N as a direct sum of cyclic R-modules.

5B. Let R be a nonzero ring with unity.
  i) Show that $R \otimes_{\mathbb{Z}} \mathbb{Z}[x] \simeq R[x]$.
  ii) Show that there is a natural injective map $R \otimes_{\mathbb{Z}} \mathbb{Z}[[x]] \to R[[x]]$ to the ring $R[[x]]$ of power series in R and that it need not be surjective.