ALGEBRA QUALIFYING EXAMINATION

JANUARY 2010

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Find a formula for the determinant of the following $n \times n$ matrix (here $x_1, x_2, \ldots, x_n$ are non-zero complex numbers):

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{x_1^{n-1}} & \frac{1}{x_2^{n-1}} & \cdots & \frac{1}{x_n^{n-1}} \end{pmatrix}$$

1B. Let F be field, V a finite-dimensional vector space over F, and Φ a nondegenerate symmetric F-bilinear form on V. Show that for any F-subspace W of W the following equation holds:

$$\mathrm{din}(W) + \dim(W^{\perp}) = \dim(V).$$

Note that $W^{\perp} := \{v \in V | \Phi(v, w) = 0, \quad \text{for all } w \in W\}$ and Φ is nondegenerate if $V^{\perp} = \{0\}$.

2A. Prove or disprove the following assertion: if G is a finite non-cyclic abelian group then its group of automorphisms $\mathrm{Aut}(G)$ is abelian.

2B. Show that a group of order 120 can not be a simple group.

3A. Let p be a prime number. Let $R = \mathbb{Z}[x]$ be the polynomial ring in one variable x over $\mathbb{Z}$. Show, for every integer $n \geq 1$, that there exists an ideal $I_n \subset R$ such that R/I_n is a field with p^n elements.

3B. Show that a prime ideal P in $\mathbb{Z}[x]$ with $P \cap \mathbb{Z} = \{0\}$ is a principal ideal.

4A. Let $f(x) = (x^2 - 2)(x^2 - 5)(x^2 - 10) \in \mathbb{Q}[x]$ and let $K/\mathbb{Q}$ be its splitting field. Calculate the Galois group of $K/\mathbb{Q}$. List all the subfields of K.

4B. Let F be a field, and let f be a separable, irreducible polynomial in $F[x]$. Show that if K is a splitting field of f over F and $\mathrm{Gal}(K/F)$ is an abelian group, then $[K\colon F] = \deg(f)$.

5A. Suppose $R = \mathbb{F}_3[x]$ is the polynomial ring in one variable x over $\mathbb{F}_3$ the field with three elements. Exhibit all the simple modules of R (upto isomorphism).

5B. Determine the elementary divisors and invariant factors of the following matrix $A \in \mathbb{F}_2[x]^{3\times 3}$ (Justify your steps; here $\mathbb{F}_2$ is the field with two elements):

$$A = \begin{pmatrix} x^2 & x^2+x+1 & x^3 \\ x^2+x+1 & x^2 & x^3+x^2+x \\ x+1 & x+1 & x^2+1 \end{pmatrix}$$