ALGEBRA QUALIFYING EXAMINATION

JANUARY 2013

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say.

1A. Let $A \in M_n(F)$ be an $n \times n$ matrix with entries in an algebraically closed field F, and let $V \subseteq M_n(F)$ be the subset of matrices that commute with A. Prove that V is an F-vector space of dimension $\geq n$.

1B. Let F be the field with two elements, and let $M = (m_{i,j})_{1 \leq i,j, \leq n}$ be the $n \times n$ matrix with $m_{i,j} = 1_F$ for all $i, j = 1, \ldots, n$. Determine the Jordan canonical form of J.

2A. Let $p > 0$ be a prime and G a finite, simple group of order that is divisible by p^2. Prove that every proper subgroup of G has index at least $2p$.

2B. Let G be a group. Show that if $\mathrm{Aut}(G)$ is a cyclic group, then G is abelian.
Hint: Consider the inner automorphism group.

3A. Let $R := M_2(\mathbb{Q})$ be the ring of 2×2 matrices with entries in $\mathbb{Q}$.
i) Exhibit a nonzero, proper left ideal of R.
ii) Prove that R is simple, *i.e.* that R has no nonzero, proper two-sided ideals.

3B. Let R be a ring with unity.
i) If R is commutative, show that the set of nilpotent elements of R is an ideal in R.
ii) Prove or disprove: If R is arbitrary, then the set of nilpotent elments is an ideal.

4A. Determine, with proof, the number of distinct roots of $x^{35} - 1$ in the field with 64 elements.

4B. Suppose F, K, and L are fields with $F \subseteq K \subseteq L$ and $[L\colon F]$ is finite. Either prove (using one of the equivalent definitions of Galois) or disprove (by exhibiting a counterexample) each of the following three assertions:
i) If L is Galois over F, then L is Galois over K.
ii) If L is Galois over F, then K is Galois over F.
iii) If L is Galois over K and K is Galois over F, then L is Galois over F.

5A. Let F be a field and set $R := F[x]$. Viewing R as a module over itself via left multiplication, let $M := R^3 = R \oplus R \oplus R$, and let N be the R-submodule of M generated by (x^2, x^3, x^4) and $(1, x+1, x^2)$. Express the quotient M/N explicitly as a direct sum of cyclic R-modules.

5B. Let $m, n \geq 1$. Describe the ring $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. In particular, what is the cardiniality of this ring?