Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers.

1A.

(a) Suppose $A$ is an $n \times n$ matrix with real entries such that all eigenvalues of $A$ are positive. Prove that $A + I$ must be invertible.

(b) ) Let $t \in \mathbb{R}$ such that t is not an integer multiple of $\pi$. For the matrix

$$A = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix},$$

prove that there does not exist a real matrix $B$ such that $BAB^{-1}$ is a diagonal matrix.

1B. Prove the following.

a) Let $L : V \to V$ be a linear map on a vector space $V$ over the field of real numbers. Suppose that $L$ is nilpotent (i.e., $L^k = 0$ for some positive integer $k$). Show that $M := I - L$ is invertible by finding an explicit formula for $(I - L)^{-1}$.

b) Let $\lambda \in \mathbb{R} \setminus \{0\}$. For the matrix

$$A = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix},$$

prove that there does not exist a real matrix $B$ such that $BAB^{-1}$ is a diagonal matrix.

2A. Let $H$ be a subgroup of a finite group $G$ if index $p$, where $p$ is the smallest prime number dividing $|G|$. Prove that $H$ is normal in $G$. (Hint: Consider the homomorphism $G \to A(S)$ induced from the action of $G$ on the cosets $S := G/H$, where $A(S)$ is the group of permutations of $S$. If $K$ is the kernel of $G \to A(S)$, then $G/K$ is isomorphic to a subgroup of the symmetric group.)

2B. Prove the following.

(a) A solvable simple group is Abelian.

(b) A simple abelian group is finite and has prime order.

(c) A solvable group with a composition series is finite.

3A. Let $D = Z[\sqrt{21}] = \{m + n\sqrt{21} | m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of $D$. Show that $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$, and conclude that $D$ is not a unique factorization domain.

3B. Prove that $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain.

4A. Show that the Galois group of $(x^2 - 2)(x^2 + 2)$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4B. Let F be a finite, normal extension of $\mathbb{Q}$ for which $|\mathrm{Gal}(F/\mathbb{Q})| = 8$ and each nonidentity element of $\mathrm{Gal}(F/\mathbb{Q})$ has order 2. Find the number of subfields of $F$ that have degree 4 over $\mathbb{Q}$ and provide a proof for your answer.

5A. Let $N$ be a submodule of a free finitely generated module $F$ over a PID $R$. Show that $N$ is a direct summand of $F$ if and only if $N \cap aF = aN$ for all $a \in R$.

5B. Suppose that $R$ is a ring with 1 and $P$ is a unitary $R$-module (i.e., $1 \cdot m = m$ for all $m \in P$). The module $P$ is called projective if for any surjective homomorphism $g : M \to N$ of $R$-modules and any homomorphism $f : P \to N$ of $R$-modules, there exists a homomorphism $h : P \to M$ of $R$-modules such that $f = gh$.

(a) Show that an $R$-module $P$ is projective if and only if $P$ is direct summand of some free module $F$.

(b) Suppose $R$ is a PID. Show that every projective $R$-module is free.