ALGEBRA QUALIFYING EXAMINATION

SPRING 2024

1A. For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( ||x|| = \sqrt{\sum_{i=1}^{n} x_i^2} \). For a linear map \( A: \mathbb{R}^n \rightarrow \mathbb{R}^n \), set \( ||A|| = \sup_{||x||=1} ||Ax|| \) (the supremum is taken over all \( x \in \mathbb{R}^n \) such that \( ||x|| = 1 \)). Suppose that \( ||A|| < 1 \). Prove that \( A + I \), where \( I \) is the identity map, is invertible.

1B. Let

\[
M = \begin{pmatrix}
4 & 2 & -2 & 5 & -1 \\
0 & 4 & 0 & 2 & 3 \\
0 & 0 & 4 & 2 & 3 \\
0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

be a matrix over \( \mathbb{C} \). Find the Jordan canonical form of \( M \).

2A. Recall that the exponent of a finite group \( G \) is the minimal positive integer \( k \) such that \( x^k = e \) for all \( x \in G \). Suppose a group \( G \) has order 12 and exponent 12. Prove that \( G \) has a subgroup of index 2.

2B. Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \), and \( P \) a Sylow \( p \)-subgroup of \( G \). Prove that \( P \cap H \) is a Sylow \( p \)-subgroup of \( H \).

3A. Suppose \( \mathbb{F} \) is a field, and \( p \in \mathbb{F}[x] \) is a degree \( n \) polynomial which has \( n \) distinct roots in \( \mathbb{F} \). Prove that the ring \( \mathbb{F}[x]/(p) \) is isomorphic to \( \mathbb{F}^n = \mathbb{F} \oplus \cdots \oplus \mathbb{F} \).

3B. Let \( R \) be a commutative ring with 1. Prove that if \( R[x] \) is a PID, then \( R \) is a field.

4A. Consider the polynomial \( p(x) = x^4 - 3x^2 + 3 \).
   (a) Let \( \pm \alpha, \pm \beta \) be its roots. Calculate \( \alpha^2 \beta^2 (\alpha^2 - \beta^2)^2 \).
   (b) Prove that the Galois group of \( p(x) \) over \( \mathbb{Q}[\sqrt{-1}] \) is cyclic.

4B. Let \( K = F(\alpha) \) be a Galois extension of \( F \), with \( \alpha \not\in F \). Suppose there exists \( \sigma \in \text{Gal}(K/F) \) such that \( \sigma(\alpha) = \alpha^{-1} \). Prove that the degree of the extension \( [K:F] \) is even and \( [F(\alpha + \alpha^{-1}):F] = \frac{1}{2}[K:F] \).

5A. Find invariant factors of the \( \mathbb{Z} \)-module \( M = (\mathbb{Z}^2 \oplus \mathbb{Z}_6) \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}_4) \), i.e. integers \( d_1 | \cdots | d_n \) such that \( M \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_n) \).

5B. Let \( R \) be a commutative ring, and \( M \) be a Noetherian \( R \)-module. Suppose \( f: M \rightarrow M \) is a surjective homomorphism. Prove that \( f \) is an isomorphism.