ANALYSIS QUALIFYING EXAM

FALL 2010

Please show all of your work. GOOD LUCK!

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a non-negative, measurable function on X. For each $E \in \mathcal{M}$, set

$$\nu(E) = \int_E f(x) d\mu(x) .$$

Prove that ν is a measure on $\mathcal{M}$ and calculate (in terms of μ)

$$\int_X g(x) d\nu(x) ,$$

for any non-negative, measurable function g.

(2) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f : X \to [0, \infty]$ be a measurable function. Show that

$$\lim_{n\to\infty} \int_X e^{-nf(x)} d\mu(x) = \mu\left(f^{-1}(\{0\})\right) .$$

Is the same true if the assumption $\mu(X) < \infty$ is removed? Prove this or provide a counterexample.

(3) Let $f : [0, \infty) \to \mathbb{R}$ be of compact support and have a continuous first derivative. Prove that for any $0 < a < b < \infty$,

$$\int_0^\infty \frac{f(bx) - f(ax)}{x} dx = -f(0) \ln(b/a)$$

holds.

(4) Let f_n be a sequence of continuous functions on $[-1.1]$. Suppose f_n converges uniformly to f on $[-1, 1]$. Prove that

$$\lim_{n\to\infty} f_n(1/n) = f(0) .$$

If the continuous functions f_n only converge to f pointwise, is the above statement still true? Prove or give a counterexample.

(5) Suppose that w is a measurable function on $\mathbb{R}^d$ with $0 < w(x) < \infty$ for almost every x in $\mathbb{R}^d$ with respect to Lebsegue measure. Suppose that K is measurable as a function on $\mathbb{R}^{2d}$ and there exists a number $A > 0$ such that:

$$\int_{\mathbb{R}^d} |K(x,y)|w(y)dy \leq Aw(x) \quad \text{for a.e. } x \in \mathbb{R}^d\,,$$

and

$$\int_{\mathbb{R}^d} |K(x,y)|w(x)dx \leq Aw(y) \quad \text{for a.e. } y \in \mathbb{R}^d\,.$$

Prove that the integral operator

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy$$

is bounded on $L^2(\mathbb{R}^d)$ with $\|T\| \leq A$.

(6) The Cantor set is the set of all $x \in [0,1]$ that can be written as

$$x = \sum_{j=1}^{\infty} c_j 3^{-j} \quad \text{with } c_j = 0, 2.$$

The Cantor function $c : [0,1] \to \mathbb{R}$ is a continuous function defined by setting

$$c(x) = \sum_{j=1}^{\infty} c_j 2^{-j-1}$$

for each $x \in C$ and declaring that c is constant on each interval from $[0,1] \setminus C$. Evaluate

$$\int_0^1 c(x)dx.$$