ANALYSIS QUALIFYING EXAM

PLEASE SHOW ALL YOUR WORK

GOOD LUCK!

PROBLEM 1

Let

$$D = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < x^\alpha\}.$$

For what values of $\alpha > 0$ is the function

$$f(x, y) = \frac{1}{(x + y)^3}$$

integrable in D?

PROBLEM 2

Do one of the following two problems:
2A. Find the following limit:

$$\lim_{\epsilon \to 0} \int_0^\infty \frac{\sin x}{x} \arctan\left(\frac{x}{\epsilon}\right) dx.$$

The integral is an improper Riemann integral. Justify all your steps.
2B. Find the following limit:

$$\lim_{n \to \infty} n^2 \int_0^{2n} e^{-n|x-n|} \log\left[1 + \frac{1}{x+1}\right] dx$$

Justify all your steps. (Hint: The substitution $y = n(x - n)$ may be useful.)

PROBLEM 3

Do one of the following two problems:
3A. Let $f(x)$ be a non-negative measurable function. Assume that the function $xf^2(x)$ is integrable on the whole real line. Prove that the function x is integrable on the set $M_y = \{x : f(x) > y\}$ for every $y > 0$, and that

$$\int_{\mathbb{R}} x f^2(x) dx = 2 \int_0^\infty y \alpha(y) dy$$

where

$$\alpha(y) = \int_{M_y} x dx.$$

3B. (a) Show that the function $f(x) = \sin(x^2)$ is not in $L^1([0,\infty))$.
(b) Show that the improper Riemann integral

$$\int_0^\infty \sin(x^2)dx$$

exists, and it equals

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt}{1+t^4}.$$

Hint: You may consider

$$\int_0^\infty \left(\int_{-\infty}^{\infty} \sin(x^2) x e^{-t^2x^2} dt \right) dx.$$

Problem 4

Let $C_r[0,1]$ be the space of real-valued continuous functions on $[0,1]$, and let $\mathfrak{M}$ be the sigma-algebra of subsets of $C_r[0,1]$ that is generated by cylinder sets

$$C_{x,(\alpha,\beta)} = \{f \in C_r([0,1]) : \alpha < f(x) < \beta\}$$

where $x \in \mathbb{R}$ and $-\infty \leq \alpha < \beta \leq \infty$. Let

$$M = \{f \in C_r([0,1]) : \sup_{0 \leq x \leq 1} |f(x)| = 1\}.$$

Prove that $M \in \mathfrak{M}$.
Hint. You may try to represent M in terms of cylinder sets by using operations of taking countable unions and countable intersections.

Problem 5

Let $u(x)$ be an absolutely continuous function on the interval $[0,1]$, and let $u(0) = 0$. Prove that

$$\int_0^1 \frac{|u(x)|^2}{x^{3/2}} dx \leq 2 \int_0^1 |u'(x)|^2 dx.$$

Problem 6

a) Let H be a Hilbert space over the field of real numbers. A sphere in H is a set of the form $\{x : ||x - x_0|| = r\}$ where $x_0 \in H$ and $r > 0$. A line in H is a set of the form $\{x : x = x_1 + ty_1\}$ for some real t where $x_1, y_1 \in H$ and $y_1 \neq 0$. Prove that a line intersects a sphere at not more than two points.
b) Does the statement remain true if H is replaced by an arbitrary Banach space E over the field of real numbers? Prove or give a counter-example.