ANALYSIS QUALIFYING EXAM

FALL 2018

Please show all of your work and state any basic results from analysis which you use. GOOD LUCK!

(1) a) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers for which

$$\lim_{n\to\infty} a_n$$

exists. Prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} a_j$$

also exists, and show that the two limits are equal.

b) Give an example to show that there exists a sequence $\{b_n\}_{n\geq 1}$ of real numbers which does not converge and for which the limit of averages

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} b_j$$

does exist.

(2) The Γ function can be defined by the following integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

where, by dt, we mean Lebesgue measure on $(0,\infty)$.

a) Show that Γ is continuous for $z > 0$.

b) Prove that

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n t^{z-1}(1-\frac{t}{n})^n dt$$

(3) Let $f \in L^1(X,\mu)$ with $f \geq 0$. Show that

$$\int_X f(x)d\mu(x) = \int_0^\infty \mu(\{x : f(x) > y\})dy$$

where, by dy, we mean Lebesgue measure on $(0,\infty)$.

(4) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. For any $1 \le p < \infty$ and $1 \le r < \infty$ consider the space $L^{p,r}$ of jointly measurable functions $f : X \times Y \to \mathbb{C}$ for which

$$\|f\|_{p,r} = \left(\int_Y \left(\int_X |f(x,y)|^p \, d\mu(x) \right)^{r/p} d\nu(y) \right)^{1/r} < \infty$$

a) Show that $\|\cdot\|_{p,r}$ is a norm on the collection of equivalence classes of functions in $L^{p,r}$ which agree a.e.
b) Show that the following analogue of the Hölder inequality is true:

$$\int_Y \int_X |f(x,y)g(x,y)| d\mu(x) d\nu(y) \le \|f\|_{p,r} \|g\|_{p',r'}$$

whenever $f \in L^{p,r}$, $g \in L^{p',r'}$, and the parameters satisfy $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$.

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{\nu_n\}_{n \ge 1}$ be a sequence of finite measures on $\mathcal{M}$ with $\nu_n << \mu$ (i.e. ν_n is absolutely continuous with respect to μ) and

$$\sum_{n=1}^{\infty} \nu_n(X) < \infty \,.$$

Show that the mapping $\nu : \mathcal{M} \to [0, \infty)$ defined by setting

$$\nu(E) = \sum_{n=1}^{\infty} \nu_n(E) \quad \text{for all } E \in \mathcal{M}$$

defines a measure. Moreover, prove that $\nu << \mu$ and

$$\frac{d\nu}{d\mu} = \sum_{n=1}^{\infty} \frac{d\nu_k}{d\mu}$$

for μ-almost every $x \in X$.

(6) Show that for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin(\pi\alpha)^2}$$

Hint: Expand the function $f(x) = e^{-2\pi i \alpha x}$ in an appropriately chosen orthonormal basis of $L^2(\mathbb{T})$.