REAL ANALYSIS QUALIFYING EXAM, AUGUST 2019

Please show all of your work and state any basic results from analysis which you use.

1. Calculate the integral

$$\int_{\mathbb{R}^2} exp(-3x^2 + 4xy - 3y^2) dm_2(x,y)$$

where m_2 denotes Lebesgue measure on $\mathbb{R}^2$.

2. Find

$$\lim_{k\to\infty} k \int_0^\infty e^{-kx^2} \arctan x dx$$

Justify all steps.

3. Let X be the set of all real sequences $x = (x_1, x_2, ...)$ such that

$$|x|_X = \sup_{n\geq 1} |\sum_{j=1}^{n} x_j| < \infty$$

(a) Show that $(X, |\cdot|_X)$ is a normed vector space.

(b) Is $l^1 \subset X$? Is $X \subset l^1$? (Provide a brief explanation or counterexample)

4. Let $f \in L^2([0,1], dx)$ with $|f|_{L^2} = 1$. Let

$$\widehat{f}(n) = \int_{x=0}^{1} f(x) exp(-2\pi i n x) dx$$

be the Fourier coefficients of f. Show that for every integer $k > 0$, at most k^2 of the $\widehat{f}(n)$ can satisfy $|\widehat{f}(n)| \geq \frac{1}{k}$.

5. Let $\phi_0(x)$, $\phi_1(x)$, .. denote the result of applying the Gram-Schmidt orthonormalization process to the sequence 1, x, x^2,.., x^n, .. in the Hilbert space $L^2([0,1], dx)$.

(a) Find an explicit expression for $\phi_2(x)$.

(b) Prove that $\{\phi_n(x)\}_{n=1}^{\infty}$ is a maximal orthonormal set.

6. Suppose that $f(x), xf(x),$ and $f'(x) \in L^2(\mathbb{R})$.

Show that f is continuous and vanishes at infinity.