ANALYSIS QUALIFYING EXAMINATION, January 2009

1. Show that the function $F(x)$ defined as $x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and $F(0) = 0$ is differentiable for every x, including $x = 0$, but $F'(x)$ is not integrable on $[-1, 1]$.

2. For any $p \geq 1$ define l^p as the space of all sequences z_j, $j = 1, 2, \ldots$, of complex numbers, satisfying

$$\sum_j |z_j|^p < \infty.$$

In addition, let c_0 be the space of sequences which converge to 0.

Prove or disprove the statement:

$$\bigcup_{p \geq 1} l^p = c_0.$$

3. Let f be a Lebesgue-integrable function on $[0, b]$ and let

$$g(x) = \int_x^b \frac{f(t)}{t} \, dt$$

for $0 < x \leq b$. Prove that g is integrable on $[0, b]$ and

$$\int_0^b g(x) \, dx = \int_0^b f(t) \, dt.$$

4. Prove that

$$\lim_{b \to 0+} \int_0^\infty \frac{\sin x}{x} e^{-bx} \, dx = \lim_{N \to \infty} \int_0^N \frac{\sin x}{x} \, dx.$$

5. Let S be the set of all real numbers α, satisfying the property that there exists a constant C and a sequence $\frac{p_j}{q_j}$ of rational numbers $(p_j, q_j \in \mathrm{Z})$, $j = 1, 2, \ldots$ such that $q_j \to \infty$ and

$$\left| \alpha - \frac{p_j}{q_j} \right| < \frac{C}{q_j^3}$$

for every j. Prove that the Lebesgue measure of S is zero.

6. Prove that there does not exist a function $I \in L^1(\mathrm{R}^d)$ such that for all $f \in L^1(\mathrm{R}^d)$

$$\int_{\mathrm{R}^d} f(y) I(x - y) \, dy = f(x)$$

for almost all $x \in \mathrm{R}^d$. This is saying that no integrable function is the unit of the convolution operation.