Analysis qualifying exam - January 2014

In problems 1, 2, 3 and 5, the measure for $L^p([a,b])$ is Lebesgue measure, and the integrals are with respect to Lebesgue measure.

(1) Let $f \in L^1([0,1])$. For $x \in (0,1)$ define

$$h(x) = \int_x^1 \frac{f(t)}{t}\, dt$$

Prove that $h \in L^1([0,1])$ and

$$\int_0^1 h(x)\, dx = \int_0^1 f(x)\, dx$$

(2) Let E_n be a sequence of Lebesque measurable sets in $[0,1]$, and let k be a real number, $0 \leq k \leq 1$. Suppose that

$$\lim_{n\to\infty} m(E_n \cap [0,a]) = ka$$

for every a, $0 \leq a \leq 1$; here m is the Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_{E_n} f(x)\, dx = k \int_{[0,1]} f(x)\, dx$$

for every function $f(x) \in L^1([0,1])$.

(3) Let $f_n(x)$ be a sequence of absolutely continuous functions on $[0,1]$ with $f_n(0) = 0$. Suppose that $f_n'(x)$ converges to $g(x)$ in $L^1([0,1])$. Prove that $f_n(x)$ converges in $L^1([0,1])$, that the limit function $f(x)$ is absolutely continuous, and that $g(x) = f'(x)$ almost everywhere.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let A_n be measurable sets with $\mu(A_n) < \infty$ and $\lim_{n\to\infty} \mu(A_n) = 0$. Let $1 < q < \infty$. Define

$$g_n = [\mu(A_n)]^{-1/q} \chi_{A_n}$$

Define p by $1/p + 1/q = 1$. Prove that for $f \in L^p(X, \mathcal{M}, \mu)$

$$\lim_{n\to\infty} \int f g_n d\mu = 0$$

(5) Let $f \in L^2([-1,1])$ such that $\int_{-1}^{1} f(x)g(x)\,dx = 0$ for all continuous functions $g(x)$ such that

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = 0$$

Prove there are real constants a, b such that $f(x) = ax + b$ a.e.

(6) Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on it. Suppose that

$$\sum_{n=0}^{\infty} \int |f|^n \, d\mu < \infty$$

(a) Prove that $|f(x)| < 1$ a.e.

(b) Prove that $[1 - f(x)]^{-1}$ is integrable with respect to μ.