ANALYSIS QUALIFYING EXAM

PLEASE SHOW ALL YOUR WORK

GOOD LUCK!

PROBLEM 1

Let

$$f(x) = \int_0^x \frac{\sin t}{t^{3/2}} dt.$$

Find

$$\lim_{k\to\infty} \int_0^\infty k^{3/2} f(x) e^{-kx} dx.$$

Justify all steps.

PROBLEM 2

Let $f(x)$ be an absolutely continuous function on $[0, 1]$. Suppose that $f(0) = 0$. Prove that

$$\int_0^1 \frac{|f(x)|^3}{x^3} dx \leq \int_0^1 |f'(x)|^3 |\ln x| dx.$$

PROBLEM 3

Let x_n be a converging sequence of complex numbers. Suppose that

$$\lim_{n\to\infty} x_n = 10.$$

Let

$$y_n = \frac{1}{n^2} \sum_{k=1}^n k x_k.$$

Prove that the sequence y_n converges and find its limit.

PROBLEM 4

Let $f \in L^1(\mathbb{R})$, and let $g, h \in L^2(\mathbb{R}, m)$. Prove that

$$\int_{\mathbb{R}} |f(x) g(x-a) h(a)| dx < \infty$$

for almost every $a \in \mathbb{R}$. Here m is the Lebesgue measure.

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Problem 5

Prove that the operator A defined by the formula

$$Af(x) = \int_0^1 \frac{f(y)}{\sqrt{|x-y|}} dy$$

is a bounded operator in $L^1([0,1], m)$ and find its norm; m is the Lebesgue measure.

Problem 6

Let

$$f(x,y) = \max\{x^2 + y^2, 1\}.$$

Define a Borel measure μ on $\mathbb{R}$ by the formula

$$\mu(E) = (m \times m)(f^{-1}(E))$$

where m is the Lebesgue measure. Find the Radon-Nikodym derivative $d\mu_{ac}/dm$ where μ_{ac} is the absolute continuous part of μ.