Analysis Qualifying Exam - January 2016

PLEASE SHOW ALL YOUR WORK

Problem 1. Let $f_n(x) = \cos(nx)$ on $\mathbb{R}$. Prove that there is no subsequence f_{n_k} converging uniformly in $\mathbb{R}$.

Problem 2. Find the following limit (with proof).

$$\lim_{n\to\infty} \int_0^\infty \frac{n^2 \sin(\frac{x}{n})}{n^3 x + x(1+x^3)} dx$$

Problem 3. Let μ be a finite Borel measure on the real line such that for all x, $\mu(\{x\}) = 0$. Let

$$F(x) = \mu((-\infty, x])$$

Prove that

$$\int_{\mathbb{R}} F(x) d\mu = \frac{1}{2}[\mu(\mathbb{R})]^2$$

Problem 4. Let δ_x denote the point mass measure at x, i.e.,

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Let m be Lebesgue measure on $\mathbb{R}$, and let μ be the Borel measure $\mu = m + \delta_0 + \delta_1$. Let λ be the signed Borel measure on $\mathbb{R}$ such that for all continuously differentiable functions $f(x)$ on $\mathbb{R}$ with bounded support we have

$$\int f(x)\, d\lambda = \int_0^1 f'(x)\, x^2\, dx$$

Prove that λ is absolutely continuous with respect to μ and find the Randon-Nikodym derivative $\frac{d\lambda}{d\mu}$.

Problem 5. Let X be a bounded subset of the real line. Let $C(X)$ be the space of bounded continuous functions on X with the usual sup norm. Let

$$U = \{f \in C(X) : f(x) > 0 \text{ for all } x \in X\}.$$

Prove that U is open if and only if X is compact.

Problem 6. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Define

$$\alpha(x) = \begin{cases} x^{p+1} & \text{if } 0 \le x \le 1 \\ x^{-p+3} & \text{if } x > 1 \end{cases}$$

Show that

$$Tf(x) = x^{-3/p} \int_0^{\alpha(x)} f(t)dt$$

is a bounded linear map from $L^q((0,\infty))$ to $L^p((0,\infty))$.

Hint: Find a function $g(x)$ such that $|\int_0^{\alpha(x)} f(t)dt| \le g(x)$.