REAL ANALYSIS QUALIFYING EXAM, JANUARY 2023

Please show all of your work and state any basic results from analysis which you use.

1. Suppose that $\phi$ is an odd smooth (i.e. $C^\infty$) function on $\mathbb{R}$. Show that the function $\frac{\phi(x)}{x}$ can be extended to define a continuous function on $\mathbb{R}$. Is this extended function necessarily smooth?

2. a) Determine the values of $a$ such that $f(x, y) = (1 - xy)^{-a}$ is $m_2$-integrable on $[0, 1] \times [0, 1]$, where $m_2$ is Lebesgue measure.

b) Define $F(a) = \int_0^1 \int_0^1 (1 - xy)^{-a} dm_2(x, y)$, for $a$ such that the integrand is $m_2$-integrable. Is $F$ differentiable on this domain, and if so, what is its derivative?

3. Suppose that $F : [0, 1] \to \mathbb{R}$. Show that there is a constant $M$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $0 \leq x, y \leq 1$ iff $F$ is absolutely continuous and $|F'(x)| \leq M$ for Lebesgue almost everywhere $x$.

4. Suppose $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}, dx)$. Show that

$$\lim_{x \to \infty} \int_x^{x+1} f(t) dt = 0$$

5. Suppose that $f(x) = x$ on $[-1/2, 1/2)$ and extend $f$ periodically.

(a) Find the Fourier series of $f$ (Either the real or complex series is acceptable).

(b) Use (a) to prove Euler’s theorem

$$\sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

6. For any Lebesgue measurable function $f : [0, 1] \to \mathbb{R}$ we define, for any $t \geq 0$,

$$\rho_f(t) = m(\{x : |f(x)| \geq t\}).$$

(a) If $f \in L^p([0, 1])$ for $1 \leq p < \infty$, show that there is a constant $C$ such that $\rho_f(t) \leq \frac{C}{t^p}$.

(b) Give an example of a function $f \notin L^2([0, 1])$ for which $\rho_f(t) \leq \frac{1}{t^p}$. 

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