Geometry/Topology Qualifying Exam
August 2019

1. Compute the following integral via the residue theorem.
$$\int_0^\infty \frac{dx}{(1+x^2)^2}.$$

2. Let S^2 be the unit sphere in $\mathbb{R}^3$ oriented by outward normal, and let ω be the restriction of the form $dx \wedge dy$ to S^2. Let also $N \subset S^2$ be the northern hemisphere.

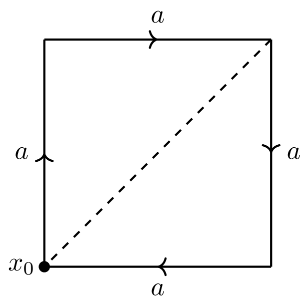
   (a) Find the integral $\int_N \omega$.

   (b) Is the form ω exact on S^2?

   Remark: The orientation of S^2 by outward normal is defined as follows. A basis (v, w) in the tangent space T_pS^2 is positively oriented if the basis (n, v, w) of $\mathbb{R}^3$ is positively oriented. Here n is the outward normal to S^2 at p.

3. Consider the group $\mathrm{O}(3)$ of orthogonal 3×3 matrices as a submanifold of the 9-dimensional space of all matrices. Consider the mapping $\mathrm{O}(3) \to \mathrm{R}^3$ which associates to every matrix $A \in \mathrm{O}(3)$ its matrix elements a_{12}, a_{13}, a_{23}. Prove that this mapping is a local diffeomorphism near the identity element.

4. Consider the space X obtained from a square $[0,1] \times [0,1]$ by identifying its sides as shown in the figure:



   Let γ be the class of the dashed path in $\pi_1(X, x_0)$. Show that $\gamma \neq \mathrm{id}$, but $\gamma^2 = \mathrm{id}$.
   Here id is the identity element in $\pi_1(X, x_0)$.

5. Let X be a solid torus, i.e. a compact region in $\mathbb{R}^3$ bounded by a torus. Consider the space Y obtained from two copies X_1, X_2 of X by identifying their boundaries via the map $(\phi, \psi) \mapsto (\psi, \phi)$, where ϕ and ψ are 2π-periodic coordinates on the torus. Let $\tilde{X}_1$ and $\tilde{X}_2$ be open sets in Y which deformation retract to X_1 and X_2, respectively.

   (a) Write down the Mayer-Vietoris sequence of reduced homology groups for the decomposition $Y = \tilde{X}_1 \cup \tilde{X}_2$.

   (b) Compute the homology groups $H_3(Y)$ and $H_0(Y)$ (with integer coefficients).

   (c) Describe explicitly the map $H_1(\tilde{X}_1 \cap \tilde{X}_2) \to H_1(\tilde{X}_1) \oplus H_1(\tilde{X}_2)$ in the above Mayer-Vietoris sequence, and show that it is an isomorphism.

   (d) Hence prove that the homology groups $H_1(Y)$ and $H_2(Y)$ are trivial.

6. Let X be a topological space, and let $Y \subset X$ be a retract of X (which means that there exists a retraction $r\colon X \to Y$). Let also k be a positive integer.

   (a) Show that $H_k(X) = 0$ implies $H_k(Y) = 0$.

   (b) Does $H_k(Y) = 0$ imply $H_k(X) = 0$? Prove your statement.