GEOMETRY/TOPOLOGY QUALIFYING EXAM

AUGUST 2023

Please show all of your work. GOOD LUCK!

(1) Find a conformal mapping from the first quadrant \( \{ z = x + iy \in \mathbb{C} : x > 0, y > 0 \} \) to the disk \( \{ z : |z| < 1 \} \).

(2) Let the mapping \( F : \mathbb{R}^4 \to \mathbb{R}^2 \) be given by the formulas \( F(x^1, x^2, x^3, x^4) = (x^1 x^2 x^3 x^4, x^1 + x^2 + x^3 + x^4) \).
   a) Prove that \( F^{-1}(1, 1) \) is a smooth submanifold of \( \mathbb{R}^4 \). What is its dimension?
   b) Find all values of \( a \) and \( b \) for which the implicit function theorem guarantees that \( F^{-1}(a, b) \) is a smooth submanifold of \( \mathbb{R}^4 \).

(3) Let \( X = S^2 / \sim \) where \( S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \} \), and the equivalence relation is \( (1, 0, 0) \sim (-1, 0, 0) ; (0, 1, 0) \sim (0, -1, 0) \). Find \( \pi_1(X) \) and \( H_2(X) \).

(4) Let \( X(x^1, x^2, x^3, x^4) = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4} \) be a vector field in \( \mathbb{R}^4 \).
   a) Show that \( X \) is tangent to the sphere \( S^3 = \{ x \in \mathbb{R}^4 : |x|^2 = 1 \} \).
   b) Let \( \phi(t) \) be the one-parameter group of diffeomorphisms of \( S^3 \) that is generated by \( X \). Compute the diffeomorphism \( \phi(\pi) \).

(5) Let \( \omega \) be a closed two-form on the four-dimensional sphere \( S^4 \).
   a) Prove that the form \( \omega \wedge \omega \) is exact.
   b) Compute \( \int_{S^4} \omega \wedge \omega \).

(6) Let \( E = \{ (z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 1 \} \).
Define a mapping \( \pi : E \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \)
by the formula \( \pi(z_1, z_2) = \frac{z_1}{z_2} \).
Prove that \( \pi \) is a covering and find its group of deck transformations.