Geometry/Topology Qualifying Exam
January 2009

1. Calculate

$$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx.$$

2. Consider the function

$$f : S^2 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to xy.$$

Find the critical points and critical values for this function.

3. Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbb{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \text{and} \quad \mathbb{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbb{X} = \frac{\partial}{\partial s}, \quad \text{and} \quad \mathbb{Y} = \frac{\partial}{\partial t}.$$

4. Let X be a connected manifold and let S^1 be the unit circle. Recall that $X \vee S^1$ is the space obtained by identifying one point in X with one point in S^1. Determine whether the S^1 is homotopically trivial in the space $X \vee S^1$.

5. The 3-ball $B^3(r) \subset \mathbb{R}^3$ is a 3-manifold with boundary $S^2(r)$, the 2-sphere of radius r. Equip $B^3(r)$ with the standard orientation and $S^2(r)$ with the induced orientation. Assume that ω is a 2-form defined on $\mathbb{R}^3 \setminus \{\vec{0}\}$ such that

$$\int_{S^2(r)} \omega = a + \frac{b}{r},$$

for all $r > 0$.

(a) Given $0 < c < d$, let $M = \{x \in \mathbb{R}^3 : c \leq |x| \leq d\}$, with standard orientation. Evaluate $\int_M d\omega$.

(b) If ω is closed, what can you say about a and b?

(c) If ω is exact in $\mathbb{R}^3 \setminus \{\vec{0}\}$, what can you say about a and b?

6. Let Γ denote the group generated by the transformations of $\mathbb{R}^2$ given by

$$A : (x, y) \to (x + 1, -y)$$

and

$$B : (x, y) \to (x, y + 1).$$

(a) Identify the surface M obtained from $\mathbb{R}^2$ by identifying (x, y) and $\gamma(x, y)$, for each $\gamma \in \Gamma$.

(b) Find explicit generators for the DeRham cohomology of the surface M (using the variables x and y).

7. Determine whether each of the following statements is true or false, and briefly explain.

(a) The tangent bundle of S^2 is a trivial vector bundle.

(b) The tangent bundle of S^3 is a trivial vector bundle.

(c) The universal covering space of $\mathbb{R}^2 \setminus \{\pm 1\}$ is contractible.

(d) If the degree of a smooth map $f : S^2 \to S^2$ is nonzero, then the map f is onto.

(e) All covering spaces of the torus $S^1 \times S^1$ are normal.