GEOMETRY/TOPOLOGY QUALIFYING EXAM, JANUARY 2010

1. For each of the following pairs of domains in the complex plane, determine whether it is possible to bijectively map one onto the other via a holomorphic (i.e. complex analytic) transformation. If so, find such a transformation, and if not, briefly explain.

a) $\{z = x + iy : 0 < x < 1\}$ and $\{0 < |w| < 1\}$.

b) $\{|z| < 1\}$ and $\{w = u + iv : v > 0\}$.

2. For the function $g : S^2 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to y^2 - z$, determine the critical points, the critical values, and qualitatively describe the level sets.

3. Consider the vector field $\vec{v}$ on $\mathbb{R}^2$ which is given by

$$\vec{v}\Big|_{\begin{pmatrix} x \\ y \end{pmatrix}} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

(and which is sometimes also written as

$$\vec{v}\Big|_{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} -y \\ x \end{pmatrix}).$$

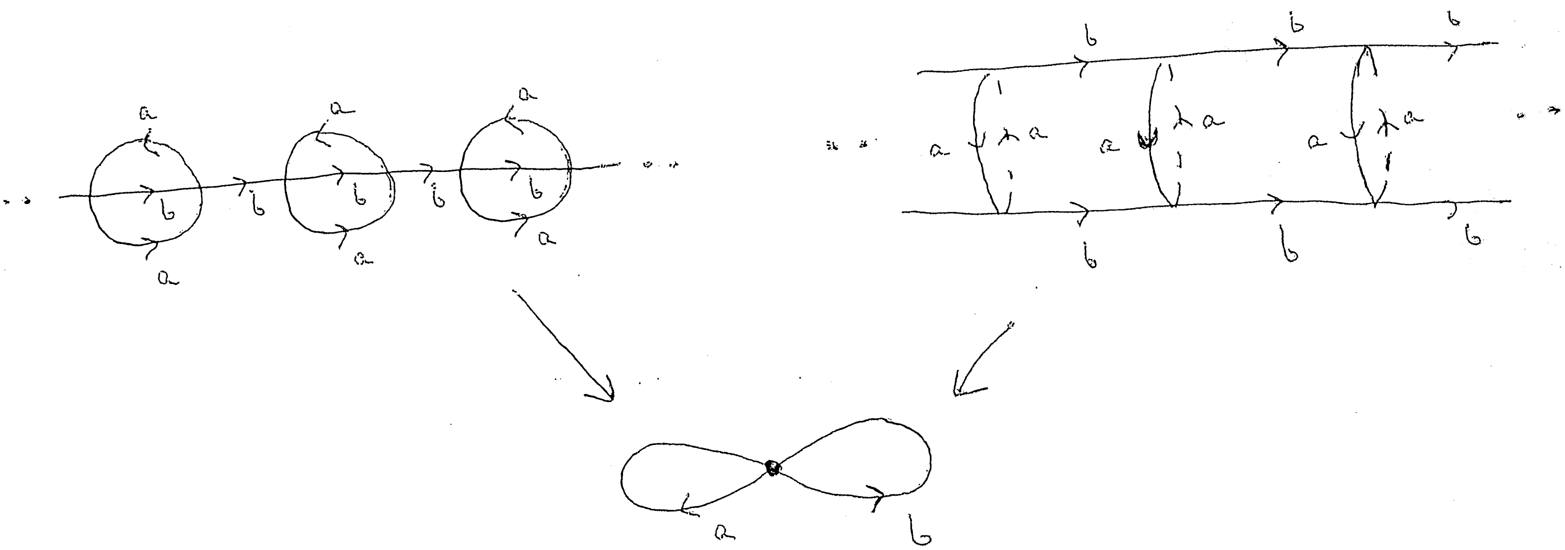
Find an explicit expression for the flow of this vector field and graph the trajectories.

4. a) Determine whether the two form $\omega = z dx \wedge dy$ is exact in $\mathbb{R}^3$.

b) Let M denote the embedded submanifold of $\mathbb{R}^3$ given by $M = \{x^2 + y^2 = 1 + z^2\}$. Determine whether the restriction of ω to M is exact.

5. Apply Mayer-Vietoris to calculate the DeRham cohomology of S^1, by considering the covering by the open sets $S^1\backslash\{N\}$ and $S^1\backslash\{S\}$, where N and S denote the north and south poles, respectively. In particular identify a generator for $H^1(S^1)$, using the connecting homomorphism.

6. Let a and b denote the standard simple loops on the figure eight, as in the drawing below. Calculate the group of automorphisms for the following covering spaces of the figure eight:*



7. Suppose that Σ is a closed embedded surface in $\mathbb{R}^3$ with outwarding pointing normal vector $\vec{n}$. Suppose also that the origin of $\mathbb{R}^3$ is contained in the bounded component of $\mathbb{R}^3 \setminus \Sigma$. Let $\vec{F}$ denote the vector field

$$\vec{F}(x,y,z) = \frac{x}{r^3}\vec{i} + \frac{y}{r^3}\vec{j} + \frac{z}{r^3}\vec{k}$$

where $r^2 = x^2 + y^2 + z^2$.

(a) Use the divergence theorem to show that the flux integral

$$\int_\Sigma \vec{F} \cdot \vec{n} dA$$

equals a flux integral integral over a small sphere S^2_ϵ (of radius ϵ) centered at the origin.

(b) Use (a) to calculate the flux integral.

* Do explain your answers.